Differential Topology Problem Set #4

Due: Tuesday, April 5

1. Consider the map
$$V \times W \to V \oplus W, \qquad v \times w \to v \oplus w.$$
Is this map bilinear?

2. Suppose $v_1, \ldots, v_p \in V$ are linearly dependent vectors. Show that
$$T(v_1, \ldots, v_p) = 0$$
for all $T \in \Lambda^p(V^*)$. Is this true for all $T \in \mathcal{J}^p(V^*)$? If so, prove it, and if not, find a counterexample.

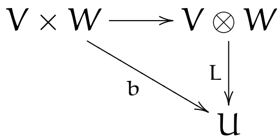
3. Chapter 10, Section 2, #3.

4. Chapter 10, Section 2, #10 (a) and (b).

5. Chapter 10, Section 3, Exercise on p. 165.

The Tensor Product of Vector Spaces. Let V and W be vector spaces over a field (you may assume they are real vector spaces). The tensor product $V \otimes W$ is a vector space equipped with a bilinear map
$$V \times W \longrightarrow V \otimes W, \qquad v \times w \to v \otimes w$$
(for every $v \in V, w \in W$) which is *universal* in the following sense. For any bilinear map
$$b : V \times W \longrightarrow U$$
where U is a vector space (over the same field), there is a unique linear map
$$L : V \otimes W \longrightarrow U$$

such that $L(v \otimes w) = b(v, w)$. In other words, any linear map $b : V \times W \longrightarrow U$ factors through the tensor product $V \otimes W$. Another way of saying this is that the diagram



commutes for every vector space U and for every bilinear map b.

6. Find a basis of $V \otimes W$ (given a basis of V and of W). Find the dimension of $V \otimes W$.

Additional problems for graduate students, or undergraduate extra credit

The n*th exterior power* of a vector space V is a vector space $\mathrm{Alt}^n V$, equipped with an alternating multilinear map

$$V \times \cdots \times V \to \mathrm{Alt}^n V, \qquad v_1 \times \cdots \times v_n \to v_1 \odot \cdots \odot v_n,$$

that is universal in the following sense. For any alternating multilinear map $b : V \times \cdots \times V \to U$ (where U is a vector space), there is a unique linear map $L : \mathrm{Alt}^n V \to U$ which takes $v_1 \odot \cdots \odot v_n$ to $b(v_1, \ldots, v_n)$.

7. Show that $\Lambda^n(V^*) \cong \mathrm{Alt}^n V^*$, where V^* is the dual to V (and $\Lambda^n(V^*)$ is the same as we defined in class). *Hint.* Show that $\Lambda^n(V^*)$ has the universality property, or use the universality property of $\mathrm{Alt}^n V^*$ to construct a map between the two spaces and then prove it's an isomorphism.

8. Show that $\Lambda^n(V)$ can be constructed as a quotient of $V \otimes V \otimes \cdots \otimes V$ (n times). In other words, there is a surjective map

$$L : V^{\otimes n} \longrightarrow \Lambda^n(V).$$

   Write down the map, show it's surjective, and find its kernel.

NOTE: The product $\odot$ is written $\wedge$ because it is in fact the same product as the one we use in class!