Differential Topology Problem Set #2

Due Thursday, Feb. 22

1. Show that X compact implies that any smooth map $f : X \to Y$ is proper. Recall that a space is called compact if, for every cover $\{U_\alpha\}$ by open sets such that

   $$X = \cup_\alpha U_\alpha$$

   there is a finite subcover. This means that there is a finite subset U_{α_i} of all the U_α such that

   $$X = \cup_{i \in I} U_{\alpha_i}$$

   and I is a finite set.

   Alternatively, you may use the definition that X is compact if and only if it is closed and bounded.

2. Chapter 1, Section 3, #2
3. Chapter 1, Section 3, #4
4. Chapter 1, Section 3, #7(a).
5. Chapter 1, Section 4, #1
6. Chapter 1, Section 4, #2(a)-(b)
7. Chapter 1, Section 4, #3
8. Chapter 1, Section 5, #1
9. Chapter 1, Section 5, #2
10. Chapter 1, Section 5, #4

Additional problems for graduate students, or undergraduate extra credit

11. Chapter 1, Section 3, 7(b)
12. Chapter 1, Section 3, #10
13. Chapter 1, Section 4, #7
14. Chapter 1, Section 4, #12
15. Suppose that X is given by

$$X = \{\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} \text{ such that } \theta_i \in \mathbb{R}\}.$$

Show that the subset

$$X' = \{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \theta \in \mathbb{R}\}$$

is a submanifold of X. Do this by finding a smooth map $f : X \to \mathbb{C}$ and showing that there is a regular value y of f such that $X = f^{-1}(y)$.

Hint. Think about determinants.