Math 639: Differential Topology Midterm Exam

Due March 22, 4:30pm

1. *(10 points)* Consider the complex curve in $\mathbb{C}^2$ defined by $\{(z, w) : w^2 - z^3 = c\}$ for some $c \in \mathbb{C}$. Under what condition(s) on c is this two-dimensional surface nonsingular? (nonsingular means "is a manifold"). *Hint:* At some point, you may want to put everything in real coordinates.

2. *(10 points)* $SO(3)$ is the set of real 3×3 matrices $A \in M(3, \mathbb{R})$ such that $AA^t = 1$ and $\det A = 1$. Prove that $SO(3)$ is a manifold by proving that it is a submanifold of $O(3) = \{A \in M(3, \mathbb{R}) : AA^t = 1\}$ (we proved that $O(3)$ is a manifold in class). Use the determinant map. *Hint:* Make sure to ask yourself what the determinant map is on $O(3)$ before you begin to reason.

3. *(10 points)* Find a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ with uncountably many critical points. Recall that "uncountably many" means infinitely many, but there is no 1-1 correspondence between the critical points and the integers. (For example, the real line has uncountably many points, but $\mathbb{Q} \subset \mathbb{R}$ is countable).

4. *(20 points)* Let $Z \subset \mathbb{R}^2$ be the unit circle. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f((x, y)) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right).$$

   (a) Prove that f is not transverse to Z.

   (b) Display a smooth homotopy $F : \mathbb{R}^2 \times I \to \mathbb{R}^2$ such that $F(x, 0) = f(x)$ and $g(x) := F(x, 1)$ is transverse to Z. Prove that g is transverse to Z.

5. *(20 points)* The space $\mathbb{R}P^2$ is defined as the set of all (one-dimensional) lines through the origin in $\mathbb{R}^3$. In this problem you will show that $\mathbb{R}P^2$ is a two-dimensional manifold.

   If a topological space M is not given via its embedding into $\mathbb{R}^m$, the way to show it is an n-dimensional (smooth) manifold is as follows. One must prove that M satisfies

- The open cover condition: for all $p \in M$, there is an open neighborhood U containing p (open in the topology on M) and a homeomorphism $\phi : U \to V$ where V is an open set of $\mathbb{R}^n$. (Recall: a homeomorphism is a continuous map whose inverse is continuous - since M didn't come with a differential structure, one can only ask that it be continuous).
- The compatibility condition: Suppose that U_1 and U_2 are two open sets around p and q, with homeomorphisms $\phi : U_1 \to V_1 \subset \mathbb{R}^n$ and $\psi : U_2 \to V_2 \subset \mathbb{R}^n$, respectively. If $U_1 \cap U_2 \neq \emptyset$, then $\phi \circ \psi^{-1} : \psi(U_1 \cap U_2) \to \phi(U_1 \cap U_2)$ is a *diffeomorphism* of open sets in $\mathbb{R}^n$.

The topology (open sets) of $\mathbb{RP}^2$ is defined as follows: for every nonzero $v \in \mathbb{R}^3$, there is a unique point l_v in $\mathbb{RP}^2$ given by the line spanned by v. Any open neighborhood V of v in $\mathbb{R}^3$ not containing 0 defines an open set in $\mathbb{RP}^2$ by $\{l_w : w \in V\}$. We define open sets on $\mathbb{RP}^2$ to be collections of points that arise this way.

(a) Show that $\mathbb{RP}^2$ is a two-dimensional manifold.

Hint: Choose a basis for $\mathbb{R}^3$. For every two linearly independent basis vectors, you can obtain an open set by taking the set of lines that do not lie in the span of those two basis vectors (i.e. the set of lines that intersect their span trivially). Write down this condition explicitly, and prove that these open sets cover $\mathbb{RP}^2$. Then find homeomorphisms from these open sets to open sets in $\mathbb{R}^2$ and prove that the composition condition on the intersections is satisfied.

(b) Let $f : S^2 \to S^2$ be the antipodal map. Explicitly, $f : (x, y, z) \to (-x, -y, -z)$ for every $(x, y, z) \in S^2 \subset \mathbb{R}^3$. Define the quotient space $S^2/\{\pm 1\}$ by the set of equivalence classes

$$S^2/\{\pm 1\} := \{(x, y, z) \in S^2 : (x, y, z) \sim f(x, y, z)\}$$

(where we identify antipodal points). Show that $S^2/\{\pm 1\}$ is a manifold (*Hint:* Use the map $S^2 \to S^2/\{\pm 1\}$). Then display a global diffeomorphism

$$f : \mathbb{RP}^2 \longrightarrow S^2/\{\pm 1\}.$$

(and show your map is a diffeomorphism).

6. *(20 points)* Suppose that X and Y are manifolds, and X is compact. Let $Z \subset Y$ be a closed submanifold. Prove that the set of smooth maps $f : X \to Y$ transversal to Z is stable under small perturbations (smooth homotopies).

7. *(10 points)* Prove that H^n and $\mathbb{R}^n$ are not diffeomorphic.

Extra Credit. *(10 points)* Generalize your proof for 5(a) to $\mathrm{Gr}_k(n, \mathbb{R})$, where $\mathrm{Gr}_k(n, \mathbb{R})$ is the set of k-planes through the origin in $\mathbb{R}^n$.