Differential Topology Final Exam

Due Tuesday, May 10, 4:30p.m.

There are four problems on this exam. You must do all four. Undergraduates will be graded with a different standard, but the exam itself is the same.

1. (20 points) Chapter 1, Section 6, #9 (p. 33).

2. (20 points) Let $h : \mathbb{R}^1 \to S^1$ be $h(t) = (\cos nt, \sin nt)$, where n is a positive integer. Suppose that ω is any 1-form on S^1. What is the relationship between $\int_{S^1} \omega$ and $\int_0^{2} \ h^*\omega$? *Hint:* If you understand #6 p. 176 you will be able to prove your answer.

3. (30 points) Let X and Y be compact, oriented manifolds. Prove that
$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$
In other words, the Euler characteristic of a product of manifolds is the product of their Euler characteristics.

4. (30 points) The *first homotopy group* or *fundamental group* of a manifold M, denoted $\pi_1(M)$ is defined as follows. Let x_0 be a point in M. Consider the set of maps
$$\{f : S^1 \to M, \quad f \text{ smooth}, f(1) = x_0\}, \quad \text{where} \quad S^1 = \{z \in \mathbb{C} : z = e^{2\ \ i}\ \}.$$
We say two maps f and g in this set are homotopic if there exists a smooth homotopy of maps between them that stays in the set. In other words, $f \sim g$ if there exists $F : S^1 \times I \to M$ such that $F(s, 0) = f(s)$ and $F(s, 1) = g(s)$ for all $s \in S^1$ and $F(1, t) = x_0$ for all $t \in I$. Then
$$\pi_1(M) = \{f : S^1 \to M,\ f \text{ smooth}, f(1) = x_0\}/\sim$$
(the set of maps modulo homotopy equivalence). The set $\pi_1(M)$ forms a group under concatination of maps: Let $[f], [g] \in \pi_1(M)$, then we define the product
$$[f] * [g] = [f * g]$$

to be the homotopy class of the map given by

$$f * g(e^{2\pi i\theta}) = \begin{cases} f(e^{2\pi i(2\theta)}) & \text{if } \theta \in [0, 1/2] \\ g(e^{2\pi i(2\theta - 1)}) & \text{if } \theta \in (1/2, 1) \end{cases}$$

You can think of this new map as spending the first half of time to trace out f and the second half of time to trace out g.

Prove that $\mathbb{Z}$ is a subgroup of $\pi_1(S^1)$. *Hint:* Try to find maps that you can prove live in different homotopy classes. Don't forget to show the group structure!!