Select Solutions Problem Set #9

1 Section 2.4

6(f) We first prove this for the case that $n = 1$. In this case,

$$\sum_{i=1}^{1}(2i-1)^3 = (2(1)-1)^3 = 1.$$

On the other hand $n^2(2n^2-1) = 1(2\cdot 1-1) = 1$. Thus the statement is true when $n = 1$.

Now we suppose $\sum_{i=1}^{n}(2i-1)^3 = n^2(2n^2-1)$ is true for some natural number n. We need to prove that this is true for the next number $n+1$. We compute:

$$\begin{aligned}
\sum_{i=1}^{n+1}(2i-1)^3 &= \sum_{i=1}^{n}(2i-1)^3 + (2(n+1)-1)^3 \\
&= n^2(2n^2-1) + (2(n+1)-1)^3 \text{ by our inductive hypothesis.} \\
&= 2n^4 - n^2 + (2n+1)^3 \\
&= 2n^4 - n^2 + 8n^3 + 12n^2 + 6n + 1 \\
&= 2n^4 + 8n^3 + 11n^2 + 6n + 1 \\
&= 2n^4 + 8n^3 + 11n^2 + 6n + 1 \\
&= (n+1)(2n^3 + 6n^2 + 5n + 1) \\
&= (n+1)^2(2n^2 + 4n + 1) \\
&= (n+1)^2(2(n+1)^2 - 1), \text{ as desired.}
\end{aligned}$$

7(d) We first check the base case when $n = 1$. In this case, $(n^3-n)(n+2) = 0$ and divisible by 12. We assume that for some natural number, $(n^3-n)(n+2)$ is divisible by 12. We need to prove that $((n+1)^3-(n+1))(n+1+2)$ is divisible by 12. We simplify:

$$\begin{aligned}
((n+1)^3-(n+1))(n+1+2) &= [n^3 + 3n^2 + 3n + 1 - (n+1)](n+3) \\
&= (n^3 - n + 3n^2 + 3n)(n+3) \\
&= (n^3-n)(n+3) + (3n^2+3n)(n+3) \\
&= (n^3-n)(n+2) + (n^3-n) + (3n^3 + 3n^2 + 9n^2 + 9n) \\
&= (n^3-n)(n+2) + 4n^3 + 12n^2 + 8n
\end{aligned}$$

By our inductive assumption, the first term is divisible by 12. Clearly $12n^2$ is divisible by 12. Finally, we need to show that $4n^3 + 8n$ is divisible by 12. It is obvious that it is divisible by 4 since $4n^3 + 8n = 4n(n^2 + 2)$. By Euclid's Lemma, then, we need only show that $n(n^2 + 2)$ is divisible by 3. This is checked case-by case. When n is a multiple of 3, i.e. $n = 3k$, the expression $n(n^2 + 2) = 3k(9k^2 + 2)$ is a multiple of 3. When $n = 3k + 1$, then $(n^2 + 2) = (9k^2 + 6k + 1 + 2)$, which is clearly divisible by 3. When $n = 3k + 2$, $n^2 + 2 = 9k^2 + 12k + 4 + 2$ which is also clearly a multiple of 3.

Therefore, since every term is a multiple of 3, the sum is a multiple of 3. By PMI, $(n^3 - n)(n+2)$ is divisible by 12 for every natural number n.

2 Section 2.5

6(b) We first check that this is true for small n. Let $n = 1$. Then $f_7 = 13$ and $4f_4 + f_1 = 12 + 1 = 13$, so the equality holds. We also check $f_8 = 21$ and $4f_5 + f_2 = 4(5) + 1 = 21$, as desired. These two cases form the "base cases" because the Fibonacci numbers f_n involves two terms with smaller indices f_{n-1} and f_{n-2}.

Now we may apply induction. We assume that the formula holds for numbers smaller than $n + 6$ and show it holds for $n + 6$. We do this directly:

$$\begin{aligned}
f_{n+6} &= f_{n+5} + f_{n+4} \text{ by the definition of Fibonacci numbers} \\
&= (4f_{n+2} + f_{n-1}) + (4f_{n+1} + f_{n-2}) \text{ by our inductive assumption} \\
&= 4(f_{n+2} + f_{n+1}) + f_{n-1} + f_{n-2} \\
&= 4f_{n+3} + f_n \text{ by the definition of Fibonacci numbers.}
\end{aligned}$$

This is what we wanted to show!

You may ask how to know what our inductive step was. In this case, we needed to use it was true for $n+5$ and $n+4$ in order to prove it was true for $n+6$. But the statement is only for $n+3 \geq 1$, or $n + 6 \geq 7$. For this reason we check the case for f_7 and f_8. This is sufficient, by induction, to prove it's true for f_9 and higher. Notice that the place we used the inductive assumption is in the second line, where we assumed the statement was true for $n + 5$ and $n + 4$. Since we then want to say it's true in the base case, we need to prove that when $n + 5 = 8$ and $n + 4 = 7$, we can rely on our base-case computation to be sure it is *actually* true.

3 Section 2.6

2(c) $|B - A| = |B| - |A \cap B|$. We know $|B|$ but we are not given $|A \cap B|$. We can find it by using $|A \cup B| = |A| + |B| + |A \cap B|$. We obtain $37 = 24 + 21 - |A \cap B|$, implying $|A \cap B| = 8$. We insert this into the first equation to find $|B - A| = 21 - 8 = 13$.

4(c) The combination rule will apply – once to pick the two of 5 players that are left-handed pitchers, and a second time to pick the other two players who are not left-handed.

6. We can use the three-set version of inclusion-exclusion. We can let A be the number of people who fell into the lake, B be the set of those who got poison ivy, and C be the set of those who got lost. Then the set of people for whom at least one of these things occurred is $A \cup B \cup C$. We want the complement of this set, or the people who didn't have anything happen to them. We calculate the size of $A \cup B \cup C$ and subtract from 40.

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 14 + 13 + 16 - 3 - 8 - 5 + 2 \\ &= 29. \end{aligned}$$

Therefore, only $40 - 29 = 11$ people were fortunate to avoid all three mishaps.

7(a) You have exactly 10 choices for your left shoe, and 9 for your left shoe. Since the number of choices for each step doesn't depend on *which* choice you make, there are $10 \cdot 9 = 90$ choices of pairs.