Select Solutions Problem Set #8

1 Section 2.4

5(g). The base case: $\prod_{i=1}^{1} x_i = x_1$.
The inductive step: $\prod_{i=1}^{1} nx_i = \prod_{i=1}^{n-1} x_i \cdot x_n$.

6(c). (1) We verify the base case when $n = 1$ by noting that $\sum_{i=1}^{1} 2^i = 2^1 = 2^2 - 2$.
(2) We now do the inductive step. We assume that, for some natural number n, $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$. We would like to prove that $\sum_{i=1}^{n+1} 2^i = 2^{n+2} - 2$. We calculate directly:

$$\sum_{i=1}^{n+1} 2^i = \sum_{i=1}^{n} 2^i + 2^{n+1} = 2^{n+1} - 2 + 2^{n+1} \text{ by the inductive step.}$$

This simplifies to $2(2^{n+1}) - 2 = 2^{n+1} - 2$, as desired.
By the PMI, $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$ for al n.

7(a). We first check the base case. Clearly when $n = 1$, $1^3 + 5(1) + 6 = 12$ is divisible by 3. Now we assume that, for some natural number n, $n^3 + 5n + 6$ is divisible by 3. We want to show this is true for $(n+1)^3 + 5(n+1) + 6$. We note that

$$\begin{aligned}(n+1)^3 + 5(n+1) + 6 &= (n^3 + 3n^2 + 3n + 1) + 5n + 5 + 6 \\ &= (n^3 + 5n + 6) + 3n^2 + 3n + 6.\end{aligned}$$

The first term in this sum is divisible by 3 by our inductive assumption. Obviously the remaining terms are also divisible by 3. Therefore the sum is divisible by 3. By PMI, $n^3 + 5n + 6$ is divisible by 3 for all natural numbers n.

2 Section 2.5

2. We assume that for all $a_k = 5a_{k-1} - 6a_{k-2} = 2^k$ for all natural numbers k less than n. We need to show this is true for $k = n$. (In other words, if S is the set of integers k for which the statement holds, then if $\{1, 2, \ldots, n-1\} \subset S$ implies $n \in S$.). The "base case" here can be verified: We note that $a_1 = 2 = 2^1$ holds. Similarly, $a_2 = 4 = 2^2$ holds. Therefore, $\{1, 2\} \subset S$. Now we assume that $a_k = 2^k$ for $k \in \{1, 2, \ldots, n-1\}$. Then

$$a_n = 5a_{n-1} - 5a_{n-2} = 5(2^{n-1}) - 6(2^{n-2})$$

by our inductive hypothesis, since the statement holds by assumption for $n - 1$ and $n - 2$. We simplify to obtain $10(2^{n-2}) - 6(2^{n-2}) = 4(2^{n-2}) = 2^n$, as desired.