Select Solutions Problem Set #13

1 Section 4.6

5(b) We show this sequence $x_n = \frac{n+1}{n}$ converges to 1. Pick $\epsilon > 0$. We need to find N such that for all $n > N$, $|x_n - 1| < \epsilon$. To that end, we note that

$$|x_n - 1| = \frac{n+1}{n} - 1 = \frac{n+1}{n} - \frac{n}{n} = \frac{1}{n}.$$

Therefore to show that $|x_n - 1| < \epsilon$, we need only show that $\frac{1}{n} < \epsilon$ for n big enough. Pick $N > \frac{1}{\epsilon}$. Then $\epsilon > \frac{1}{N}$ (multiply both sides by ϵ and divide both sides by N). Then if $n > N$, we have $\frac{1}{n} < \frac{1}{N} < \epsilon$. Therefore, $|x_n - 1| = \frac{1}{n} < \epsilon$, as desired.

5(l) We show that $x_n = \left(\frac{n}{2}\right)^n$ diverges. You could show that it gets arbitrarily large (and therefore cannot converge to a finite value). Alternatively, you can assume it converges and find a contradiction. We'll go this second route.

For any natural number M with $M > 1$ and for all natural numbers N, we show there exists $n > N$ such that $x_n > M$. Pick $n = kM$, for any $k \in \mathbb{N}$ chosen so that $kM > N$. Then

$$x_n = \left(\frac{kM}{2}\right)^{kM} = \left(\frac{k}{2}\right)^{kM} M^{kM},$$

which is clearly greater than M as long as $M > 1$, as it is by assumption. Therefore $x_n > M$ for $n > kM$.

Now suppose the sequence converges to some value L. Then pick some $\epsilon > 0$, say $\epsilon = .8$. Since the sequence converges, there exists N such that $|x_n - L| < .8$ if $n > N$. However, we just showed that there exists some $n > N$ such that $x_n > L + 1$, which implies there exists $n > N$ such that $|x_n - l| > 1$, contrary to assumption.

2 Section 5.2

2(a) We suppose the set is finite, and find a contradiction. Suppose that $\mathbb{N} - \mathbb{N}_{15}$ were finite. Then there is a bijection

$$f : \mathbb{N}_k \longrightarrow \mathbb{N} - \mathbb{N}_{15}$$

for some k. *We will show that* f *cannot be surjective.*. Let $n = max\{f(1), f(2), \dots, f(k)\} + 1$. First, we note that n is not an element of $\mathbb{N}_{15}$ because $n > f(1)$ and $f(1) > 15$, by assumption (note the codomain of f). Furthermore, there is no element $x \in \mathbb{N}_k$ such that $f(x) = n$, since n is bigger than the images $f(1), \dots, f(k)$. Therefore, f is not onto.

3 Section 5.3

5(a) We did this problem in class. However, here is yet another proof! Suppose that A is denumerable, and $x \in A$. Then there exists a bijection $f : \mathbb{N} \mapsto A$. Then since f is surjective, there exists some $n \in \mathbb{N}$ such that $f(n) = x$. Consider the map $g : \mathbb{N} \mapsto A - \{x\}$, given by

$$g(k) = \begin{cases} f(k) \text{ if } k < n \\ f(k+1) \text{ if } k \geq n \end{cases}$$

for any $k \in \mathbb{N}$. Notice that g is defined for all integers (even for n), but its image does not include $x = f(n)$. We now show that g is a bijection. It is 1-1 because f is 1-1, [Suppose $g(a) = g(b)$. If $a, b < k$, we have $f(a) = f(b)$, which implies $a = b$. If $a, b \geq k$, then $g(a) = g(b)$ implies $f(a+1) = f(b+1)$, which implies $a+1 = b+1$ since f is 1-1, and thus $a = b$. Finally, if $a < k$ and $b \geq k$ and $g(a) = g(b)$, then $g(a) = f(a) = g(b) = f(b+1)$, implies $a = b+1$ since f is 1-1. This implies $a \geq b$, contrary to assumption.]. The map g is also surjective, which we can see as follows. Pick any $a \in A - \{x\}$. There exists some $k \in \mathbb{N}$ such that $f(k) = a$, since f is onto. If $k < n$, then $f(k) = g(k) = a$, so a is in the image of g. If $k \geq n$, then $a = f(k) = g(k-1)$, so a is in the image of g as well. This implies that g is onto. Therefore g is a bijection, proving that $A - \{x\}$ is denumerable.

9(f) We may write the elements of this set in a matrix format, with columns indicated by n and rows indicated by k. In this case, the i, jth entry of the matrix is $i/2^j$. Some of these numbers will be reducible fractions – cross those off the list of elements in the column. Now, create a bijection using the same method as we did in class for $\mathbb{Q}$, mainly draw diagonal lines from the top down to the left, each time "numbering" the entries that you encounter as you draw these diagonal lines. This numbering indicates a bijection of this set with $\mathbb{N}$.