MATH 290-001 Midterm Exam 2 Solutions

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Please write your work carefully in the space provided. You are welcome to rewrite your proofs, but if you do so please circle the part you would like to submit as your answer. You will be graded for proof style as well as for content, so please be careful to write in complete sentences with correct grammar and punctuation. There are 10 problems, totaling 50 points.

1. Consider the family $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$, where $A_i = \left[i, 2^i\right] \subset \mathbb{R}$.

   (a) *(2 points)* Are the sets in $\mathcal{A}$ pairwise disjoint? Justify your answer.

   These sets are not pairwise disjoint, because there exist pairs of sets that do intersect nontrivially. For example, A_1 and A_2 both contain 2, so the sets are not disjoint.

   (b) *(3 points)* Prove that

$$\bigcap_{i=1}^{\infty} A_i = \emptyset$$

Pick $x \in \bigcap_{i=1}^{\infty} A_i$. Since $x \in A_1 = [1, 2]$, we have $x \leq 2$. On the other hand, since $x \in A_3 = [3, 8]$, we know $x \geq 3$. This is a contradiction, so there are no points in the intersection.

2. *(5 points)* Use the principle of mathematical induction to prove the statement: The sum of the first n odd numbers is n^2. *Hint:* the way to write this statement mathematically is:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

We first check the base case. Certainly when $n = 1$, we have $2(1) - 1 = 1^2$, so the statement is true.

Now assume the statement is true for n. We show it is true for $n + 1$.

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2n - 1) + [2(n + 1) - 1] &= [1 + 3 + 5 + \cdots (2n - 1)] + 2(n + 1) - 1 \\ &= n^2 + 2(n + 1) - 1 \text{ by the inductive assumption.} \\ &= n^2 + 2n + 1 = (n + 1)^2. \end{aligned}$$

By the principal of mathematical induction, the statement is true.

3. *(5 points)* Use the generalized principle of mathematical induction to prove the statement that $2^n > n^2 - 2$ for all natural numbers $n > 2$.

   We first check the base case. When $n = 3$, the statement is that $2^3 = 8$ is greater than $3^2 - 2 = 7$, so the statement is true.

   Now we assume it is true for n and we show that $2^{n+1} > (n+1)^2 - 2$. First, we notice that $(n+1)^2 - 2 = n^2 + 2n - 1$. We observe:

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n = 2^n + 2^n \\ &= (n^2 - 2) + 2^n \text{ by inductive assumption.} \end{aligned}$$

   We note that $2^n > 2n + 1$ for $n \geq 3$, since $2^n = 2 \cdot 2^{n-1} > 2 \cdot 2^{n-2} + 1 > 2n + 1$ when $n \geq 3$.

   Therefore $2^{n+1} > (n^2 - 2) + 2^n > (n^2 - 2) + (2n + 1) = (n+1)^2 - 2$, as desired.

4. *(5 points)* Prove that If $a > 0$, then for every natural number n, $a^n > 0$. *Hint.* Use the well-ordering principle. You can earn 2 points by stating the principle.

   The well-ordering principle states that any non-trivial subset of $\mathbb{N}$ has a least element. Let $a > 0$ be any real number, and

   $$S = \{n \in \mathbb{N} : a^n \leq 0\}.$$

   We show this subset is empty, proving that $a^n > 0$ for all natural numbers. Suppose S it is not empty. Then there is a least element $m \in S$ by the well-ordering principle, and $a^m \leq 0$. Since $a \neq 0$, we divide by a and obtain $a^{m-1} \leq 0$. Since $m \in \mathbb{N}$, $m - 1 \geq 0$. If $m - 1 = 0$, then $a^{m-1} = 1$, a contradiction. So $m - 1 > 0$, But then $m - 1 \in S$ since $a^{m-1} \leq 0$. This contradicts the statement that m is the *least* element of S. Therefore, S must be empty, proving the statement that $a^n > 0$ for each $n \in \mathrm{N}$.

5. *(5 points)* A class of GMU students consists of 10 people from Virginia, 6 people from New York, and 5 people from California. How many ways can a group be formed from these students with exactly 3 people form Virginia, two from New York and two from California?

   We must choose 3 students out of 10 from Virginia. There are $\binom{10}{3}$ such choices (10 choose 3). We must choose 2 students out of 6 from New York – there are $\binom{6}{2}$ such choices. We must choose two students from 5 in California; there are $\binom{5}{2}$ of these. The choices of each of these sets is independent of the choices made for the other sets. Therefore, by the multiplication principle, the choice of a group with 3 from VA, 2 from NY and 2 from CA is the product:

$$\binom{10}{3} \cdot \binom{6}{2} \cdot \binom{5}{2} = 18{,}000$$

6. *(5 points)* A group of 30 people go to a night club together. Fifteen people order appetizers, 10 people order drinks, and 12 people make music requests. Eight people order drinks and also make music requests, 7 people order appetizers and drinks, and 6 people order appetizers and make music requests. How many people order appetizers and drinks and also make music requests? Be sure to show your work for credit.

   NOT GRADED!

7. *(5 points)* Let $A = \{1, 2, 3, 4, 5\}$ and R and S be relations on A given by $R = \{(x, y) : x < y\}$ and $S = \{(x, y) : y < x^2 - 2\}$. Find $S \circ R$. First we write down $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4),$ Then we write down $S = \{(2, 1), (3, 1), (4, 1), (5, 1), (3, 2), (4, 2), (5, 2), (3, 3), (4, 3), (5, 3), (3, 4), (4, 4), (5, 4), (3,$ In each case, these are the pairs that satisfy the relations stated. Check! FInally, we find $S \circ R$ by noting that these are the pairs in (i, j) where $(i, k) \in R$ and $(k, j) \in S$. (For example, since $(1, 5) \in R$ and $(5, 4) \in S$, we conclude $(1, 4) \in S \circ R$. All together we obtain

$$S \circ R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5)\}.$$

(Note that $(5, k)$ is not in the composition for any k because 5 is not in the first entry of any elements of R.)

8. *(6 points)* Let R be the relation on $\mathbb{R}$ that xRy if and only if $x = y$ or $xy = 1$. Prove that R is an equivalence relation.

We need to show that R is reflexive, symmetric, and transitive. R is reflexive because $x = x$ implies xRx. R is symmetric because xRy implies $x = y$ or $xy = 1$, which implies $y = x$ or $yx = 1$, implying yRx. Finally, we check that R is transitive. Assume that xRy and yRz. Then($x = y$ or $xy = 1$) and ($y = z$ or $yz = 1$). We check all cases: If $x = y$ and $y = z$ then clearly $x = z$, implying xRz. If $x = y$ and $yz = 1$, then by substituting for y, we obtain $xz = 1$, which implies xRz. Similarly, if $xy = 1$ and $y = z$ then we can substitute for y in the first equation and obtain $xz = 1$, which implies xRz. Finally, if $xy = 1$ and $yz = 1$, then we find $y = 1/z$ which we substitute into the first equation and get $x = z$, which implies xRz.

9. *(4 points)* Consider the integers under the equivalence relation *mod* n. How many equivalence classes are there? Justify your answer.

   There are n equivalence classes, given by $[0], [1], [2], \ldots, [n-1]$ where each set $[k] = \{\ldots, k-2n, k-n, k, k+n, k+2n, \ldots\}$. There are n of them because any number greater than or equal to n, or less than 0, is equal mod n to one of the classes already listed. Conversely, the classes listed do not overlap because the number $0, 1, 2, \ldots, n-1$ differ by less than n.

10. *(5 points)* Write down the partition of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ given by by equivalence relation xRy iff $x + y = 11$.

    $\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}$