MATH 290-001 Midterm Exam 1

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Please write your work carefully in the space provided. You are welcome to rewrite your proofs, but if you do so please circle the part you would like to submit as your answer. You will be graded for proof style as well as for content, so please be careful to write in complete sentences with correct grammar and punctuation. There are 14 problems, totally 50 points.

1. *(2 points)* State the contrapositive of the statement: If my checking account balance is over $300, I have a good job and I am saving money.

2. *(3 points)* Prove this version of De Morgan's Law: For propositions P and Q,

$$\sim(P \vee Q) \Leftrightarrow (\sim P \wedge \sim Q).$$

3. *(3 points)* Choose the English statement and explanation that corresponds with the logical phrase
$$(\forall n \in \mathbb{N})(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x^n > y).$$
Do not be concerned with the truth or falsehood of the statement!

   (a) For each natural number n, there is a real number x so that $x^n > y$ for any real number y. In particular, for any specific n, one value of x will make the statement $x^n > y$ hold for every possible value of y.

   (b) For each natural number n, and for each values of y, there is some value of x such that $x^n > y$. In particular, for any specific n, the value of x that makes the statement $x^n > y$ true may depend on y.

   (c) There exists some value of x together with all possible values of y and a natural number n, so that $x^n > y$ holds.

   (d) There exists a real number x so that $x^n > y$ for all natural numbers n and all values $y \in \mathbb{R}$.

   (e) For every real number y and every natural number n, there is a real number x that makes $x^n > y$ true.

   Answer: ___________________________

4. *(3 points)* Prove the statement above, or give a counterexample.

5. *(2 points)* State (any version of) Euclid's lemma.

6. *(2 points)* Let the universe be $\mathbb{Z}$, the set of integers. Let E be the set of even integers, and $\mathbb{Z}^-$ be the set of negative integers. Find $(E \cap \mathbb{Z}^-)^c$.

7. *(4 points)* Prove that $\sqrt{5}$ is not rational.

8. *(7 points)* Consider the statement: If x and y are even integers, then xy is divisible by 4.

   (a) Prove the statement using a direct proof.

   (b) State the converse.

   (c) Is the converse true? Prove it or give a counterexample.

9. *(4 points)* Let $A(x)$ be an open sentence with variable x. Prove

$$\sim(\exists x)A(x) \quad \text{is equivalent to} \quad (\forall x)\sim A(x).$$

10. *(4 points)* Prove that there are no integers m, n such that $15m + 81n = 2$.

11. *(4 points)* Let A and B be sets. Prove if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

12. *(4 points)* Let $\mathcal{A} = \{[n, n+1), n \in \mathbb{N}\}$ be a family of sets, each of which is the interval $[n, n+1)$, closed on the left and open on the right. Then the union over $\mathcal{A}$ is denoted by $\bigcup_{n=1}^{\infty}[n, n+1)$. Prove that

$$\bigcup_{n=1}^{\infty}[n, n+1) = [1, \infty).$$

13. *(4 points)* Correct the following proof:

Claim: Suppose m is an integer. If $m^2 - 2m + 1$ is even, then m is odd.

Proof: $m \in Z$ and $m^2 - 2m + 1$ even. m odd means $m = 2k + 1$.

$$m^2 - 2m + 1 = (2k+1)^2 - 2(2k+1) + 1 = 4k^2 + 4k + 1 - 4k - 2 + 1 + 1 = 4k^2 = 2(2k^2).$$

Let ℓ be an integer. Then $\ell = 2k^2k$. Thus $m^2 - 2m + 1 = 2\ell$.

Therefore if $m^2 - 2m + 1$ is even, then m is odd.

Additional Comments or Corrections:

14. *(4 points)* Write a (correct) proof of the Claim above.