Math 113: Exam 1
In place of on-line exam
Prof. Goldin

Name: KEY

You are not permitted to use any aids, such as calculators, textbooks, notes, friends, tutors, etc. for this test. You must show all your work for credit. Partial credit will be given when appropriate, but a correct answer does not guarantee full credit if work is not shown.

1. Inverse Functions:

(a) Suppose $f(x)$ is a one-to-one function on its domain, and $g(x) = f^{-1}(x)$ is its inverse. By definition, what does $f(g(x))$ equal?

$\boxed{f(g(x)) = x}$

(b) Let $f(x) = \sqrt{x+1}, \quad x \geq -1$. Find $f^{-1}(x)$.

$y = \sqrt{x+1}$

$y^2 = x+1$

$y^2 - 1 = x \qquad x \geq -1$

$\Rightarrow \quad y^2 \geq 0$

But $\quad y = \sqrt{x+1} \Rightarrow \quad y \geq 0$

$\boxed{f^{-1}(x) = x^2 - 1}$

(switch x and y)

$\boxed{\text{and } y \geq 0}$

2. Trigonometric functions: Fill in the answer.

(a) $\sin(\frac{\pi}{2}) =$ 1

(b) $\cos(\frac{\pi}{3}) =$ $\frac{1}{2}$

(c) $\tan(\frac{\pi}{4}) =$ 1

(d) $\sin^{-1}(\frac{\sqrt{3}}{2}) =$ $\frac{\pi}{3}$

3. Suppose $f(x)$ is a function defined in an open interval around $x = a$. State the definition of

$$\lim_{x \longrightarrow a} f(x) = L,$$

where L is a real number.

For all $\varepsilon > 0$, there exists $\delta > 0$

such that $|f(x) - L| < \varepsilon$

whenever $0 < |x-a| < \delta$

4. Given that

$$\lim_{x \to 2} 3x + 1 = 7$$

Find a $\delta > 0$ that works for $\epsilon = .06$.

Write what you want to prove → Find $\delta > 0$ such that $|3x+1-7| < .06$ whenever $|x-2| < \delta$

simplify → $|3x-6| < .06$

$3|x-2| < .06$

$|x-2| < .02$ ← notice this is what you want here

Let $\delta = .02$

Then $|x-2| < .02$ implies $3|x-2| < .06$

which implies $|3x+1-7| < .06$.

So $\boxed{\delta = .02 \text{ meets the } \varepsilon = .06 \text{ Challenge}}$

5. Find a parameterization of a circle of radius 3 centered at the origin.

$$\boxed{\begin{aligned} x &= 3\cos t \\ y &= 3\sin t \end{aligned} \qquad 0 \le t < 2\pi}$$

notice that $x^2+y^2 = 9\cos^2 t + 9\sin^2 t$

$= 9(\cos^2 t + \sin^2 t)$

$= 9 \cdot 1$

$= 3^2$

is a circle of radius 3.

6. Find the equation of a line tangent to the graph of the function

$$f(x) = x^2 - 4x$$

at the point $P(1, -3)$. Show all your work, especially of any limits you calculate. You may use the concept of derivative to check your work.

Slope at $a=1$ is

$$m = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 4(1+h) - [1^2 - 4(1)]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{4} - 4h - \cancel{1} + \cancel{4}}{h}$$

$$= \lim_{h\to 0} \frac{2h + h^2 - 4h}{h} = \lim_{h\to 0} \frac{h^2 - 2h}{h} = \lim_{h\to 0} h - 2 = -2$$

$\Rightarrow$ slope $= -2$.

Equation of line is $y = -2x + b$

Find y-intercept by plugging in $(1, -3)$

$$-3 = -2(1) + b \Rightarrow b = -1$$

Answer $\boxed{y = -2x - 1}$

7. Let

$$f(x) = \frac{2x^2}{x^2 - x - 2}.$$

Find equations for the horizontal asymptotes. Justify your answer.

Horizontal asymptotes ~~[illegible]~~ occur when $x \longrightarrow \pm\infty$

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-x-2} = \lim_{x\to\pm\infty} \frac{2x^2 \cdot \frac{1}{x^2}}{(x^2-x-2)\cdot\frac{1}{x^2}}$$

$$= \lim_{x\to\pm\infty} \frac{2}{1-\frac{1}{x}-\frac{2}{x}}$$

As $x \longrightarrow +\infty$ or $x \longrightarrow -\infty$, $\frac{1}{x} \to 0$ and $\frac{2}{x} \to 0$

~~[illegible]~~ So limit on top is finite and limit on bottom is finite.

Apply the quotient rule

$$= \frac{\lim_{x\to\pm\infty} 2}{\lim_{x\to\pm\infty} 1-\frac{1}{x}-\frac{2}{x}} = \frac{2}{1\pm0\pm0} = 2$$

Thus $\boxed{y=2}$ is only horizontal asymptote

8. Calculate the limit:

$$\lim_{x \longrightarrow -1} \frac{x+1}{x^2 - x - 2}$$

Notice limit on bottom is $1-(-1)-2=0$ so we can't apply quotient rule.

$$\lim_{x \to -1} \frac{x+1}{x^2-x-2} = \lim_{x \to -1} \frac{\cancel{x+1}}{\cancel{(x+1)}(x-2)}$$

$$= \lim_{x \to -1} \frac{1}{x-2} = \boxed{\frac{1}{-3}}$$

now apply quotient rule

9. Calculate the limit:

$$\lim_{x \longrightarrow 0} \frac{5x^2 + 7}{x^2}$$

Again, can't apply quotient rule since $x^2 \to 0$ as $x \to 0$

$$\lim_{x \to 0} \frac{5x^2+7}{x^2} = \lim_{x \to 0} \frac{5\cancel{x^2}}{\cancel{x^2}} + \frac{7}{x^2} = \lim_{x \to 0} 5 + \frac{7}{x^2}$$

however as $x \to 0$, $\frac{7}{x^2} \longrightarrow \infty$, from both the left and the right.

Thus $\boxed{\lim_{x \to 0} 5 + \frac{7}{x^2} = \infty}$

10. Let

$$f(x) = \begin{cases} 2x - 7 & x \geq 2 \\ 5x + 1 & x < 2 \end{cases}$$

Calculate the limit $\lim_{x \longrightarrow 2^-} f(x)$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (5x+1)$$

Since x approaches 2 from the left, we use the def'n of f(x) left of 2.

$$\boxed{\lim_{x \to 2^-} (5x+1) = 5(2)+1 = 11}$$