An Introduction to Lie Algebras and their Cohomology

Combinatorics, Algebra, & Geometry Seminar

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Warning

I am not a combinatorist, algebraist, or geometer!

By sitting in this talk, you agree not to ask any questions.

Lie Algebras

What Is A Lie Algebra?

Definition. A vector space L over a field $\Bbbk$, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [x, y]$ called the *bracket* of x and y, is called a *Lie algebra* over $\Bbbk$ if the following are satisfied:

(L1) The bracket operation is bilinear.

(L2) $[x, x] = 0$ for all $x \in L$.

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. (Jacobi identity)

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Facts.

- (L1) and (L2) imply $[x, y] = -[y, x]$ for all $x, y \in L$ (L2') (anticommutivity).
- If $\mathrm{char}(\Bbbk) \neq 2$, then (L2') implies (L2).

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- A non-abelian Lie algebra L is called *simple* if there are no non-trivial proper ideals. If L is simple, then $Z(L) = 0$ and $[L, L] = L$.
- A Lie algebra L is called *semisimple* if it is the direct sum of simple Lie algebras.

1. Let $L = \mathbb{R}^3$ and $\mathbb{k} = \mathbb{R}$ and define $[\vec{x}, \vec{y}] = \vec{x} \times \vec{y}$ for $\vec{x}, \vec{y} \in \mathbb{R}^3$. Then $\mathbb{R}^3$ is an $\mathbb{R}$-Lie algebra.

Examples of Lie Algebras

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2. Let $M_n(\mathbb{k})$ denote the $n \times n$ matrices over $\mathbb{k}$ and define $[A, B] = AB - BA$. Then $M_n(\mathbb{k})$ is a $\mathbb{k}$-Lie algebra.

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3. Let V be a $\mathbb{k}$-vector space. Then $\mathrm{End}\, V$ is a $\mathbb{k}$-Lie algebra with bracket defined by $[f, g] = f \circ g - g \circ f$. When considering $\mathrm{End}\, V$ as a Lie algebra, we denote it as $\mathfrak{gl}(V)$, called the *general linear algebra*.

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4. When V is finite dimensional over $\mathbb{k}$, we can identify $\mathfrak{gl}(V)$ with the set of $n \times n$ matrices over $\mathbb{k}$, which we denote by $\mathfrak{gl}_n(\mathbb{k})$. Using matrices makes calculations simpler.

Let G be a smooth manifold which is also a topological group with multiplication map $\mathrm{mult} : G \times G \to G$ and inverse map $\mathrm{inv} : G \to G$. Then G is a *Lie group* if mult and inv are smooth maps.

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The Lie algebra associated with the Lie group GL_n of invertible matrices is $\mathfrak{gl}_n$, the vector space M_n of square matrices.

Let V be a vector space over $\mathbb{C}$ with $\dim V = n$. The *special linear algebra* is defined as

$$A_{\ell-1} = \mathfrak{sl}_n(\mathbb{C}) = \{B \in M_n(\mathbb{C}) : \operatorname{Tr} B = 0\}.$$

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Consider the case when $n = 2$. We can write an ordered basis for $\mathfrak{sl}_2(\mathbb{C})$ as

$$\left\{ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Definition. Let $\mathfrak{A}$ be a Lie algebra over $\mathbb{k}$. A (left) $\mathfrak{A}$-*module* M is a $\mathbb{k}$-module equipped with a $\mathbb{k}$-bilinear product $\mathfrak{A} \otimes_{\mathbb{k}} M \to M$ (written $a \otimes m \mapsto a.m$) such that

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The *invariant submodule* $M^{\mathfrak{A}}$ of an $\mathfrak{A}$-module M is defined by

$$M^{\mathfrak{A}} = \{m \in M : x.m = 0 \text{ for all } x \in \mathfrak{A}\}.$$

Derivations of a Lie Algebra

Definition. Let $\mathfrak{A}$ be a $\mathbb{k}$-Lie algebra. A *derivation* of $\mathfrak{A}$ is a linear map $\delta : \mathfrak{A} \to \mathfrak{A}$ such that $\delta([a, b]) = [a, \delta(b)] + [\delta(a), b]$.

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The set of *inner derivations* of $\mathfrak{A}$ is defined as $\mathrm{Der}_{\mathrm{Inn}} = \{\mathrm{ad}\,x : x \in \mathfrak{A}\}$ and forms an ideal in $\mathrm{Der}\,\mathfrak{A}$.

Definition. A *representation* of a $\mathbb{k}$-Lie algebra $\mathfrak{A}$ is a homomorphism $\rho : \mathfrak{A} \to \mathfrak{gl}(V)$ for some $\mathbb{k}$-vector space V.

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Corollary. Any simple Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(\mathfrak{A})$.

Killing Form of a Lie Algebra

Let $\mathfrak{A}$ be a Lie algebra. If $x, y \in \mathfrak{A}$, then define

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The map κ is called the *Killing form* of $\mathfrak{A}$, and is a symmetric bilinear form on $\mathfrak{A}$.

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The Killing form is said to be *nondegenerate* if the radical of κ is 0, where

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Theorem. A Lie algebra $\mathfrak{A}$ is semisimple if and only if its Killing form is nondegenerate.

Adjoint Representation & Killing Form of $\mathfrak{sl}_2(\mathbb{C})$

By direct calculation we have

$$\operatorname{ad} e = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \operatorname{ad} h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \operatorname{ad} f = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

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Since $\det \kappa = -128 \neq 0$, $\mathfrak{sl}_2(\mathbb{C})$ is semisimple.

Cohomology of Lie Algebras

The Exterior Algebra

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If $\pi : T^k(\mathfrak{g}) \longrightarrow \Lambda^k \mathfrak{g}$ is the canonical map, then

$$\pi(a_1 \otimes \cdots \otimes a_k) = a_1 \wedge \cdots \wedge a_k.$$

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The *universal enveloping algebra* of $\mathfrak{g}$ is defined as

$$U(\mathfrak{g}) = T(\mathfrak{g})/\mathfrak{J},$$

and is an associative algebra with identity.

Construction of $H^*(\mathfrak{g}, M)$

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$$(d_n f)(g_1 \wedge \cdots \wedge g_{n+1}) = \sum_{\ell=1}^{n+1} (-1)^{\ell+1} g_\ell . f(g_1 \wedge \cdots \wedge \hat{g}_\ell \wedge \cdots \wedge g_{n+1}) + \sum_{r<s} (-1)^{r+s} f([g_r, g_s] \wedge g_1 \wedge \cdots \wedge \hat{g}_r \wedge \cdots \wedge \hat{g}_s \wedge \cdots \wedge g_{n+1}).$$

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$$(d_0 f)(g) = g.f$$
$$(d_1 f)(g_1 \wedge g_2) = g_1.f(g_2) - g_2.f(g_1) - f([g_1, g_2])$$

In the sequence

$$\cdots \xrightarrow{d_{n-1}} C^n(\mathfrak{g}, M) \xrightarrow{d_n} C^{n+1}(\mathfrak{g}, M) \xrightarrow{d_{n+1}} \cdots$$

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Define the n-*cocycles* as $Z^n(\mathfrak{g}, M) = \ker d_n$ and the n-*coboundaries* as $B^n(\mathfrak{g}, M) = \operatorname{im} d_{n-1}$. We have $B^n(\mathfrak{g}, M) \subseteq Z^n(\mathfrak{g}, M)$ by the exactness of the sequence.

Construction of $H^*(\mathfrak{g}, M)$

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We define the n^{th} *cohomology of* $\mathfrak{g}$ *with coefficients in* M by

$$H^n(\mathfrak{g}, M) = Z^n(\mathfrak{g}, M)/B^n(\mathfrak{g}, M).$$

$$C^0(\mathfrak{g}, M) = \mathrm{Hom}_{\mathbb{C}}(\Lambda^0\mathfrak{g}, M) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, M) \cong M$$
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Theorem. The 0^{th} cohomology of $\mathfrak{g}$ with coefficients in M is the invarient submodule of $\mathfrak{g}$, that is

$$H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}.$$

The 1-cocyles $Z^1(\mathfrak{g}, M)$ is the set of $f \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, M)$ such that

$$f([g_1, g_2]) = g_1.f(g_2) - g_2.f(g_1) \quad \text{for all } g_1, g_2 \in \mathfrak{g}.$$

This is exactly the set of derivations from $\mathfrak{g}$ to M, denoted $\mathrm{Der}(\mathfrak{g}, M)$.

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The 2-coboundaries $B^1(\mathfrak{g}, M)$ is the image of d_0, which is the set of elements $d_0 m$ in $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, M)$ such that $(d_0 m)g = g.m$. This is the set of inner derivations from $\mathfrak{g}$ to M, denoted $\mathrm{Der}_{\mathrm{Inn}}(\mathfrak{g}, M)$.

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Theorem. The 1^{st} cohomology of $\mathfrak{g}$ with coefficients in M is

$$H^1(\mathfrak{g}, M) = \mathrm{Der}(\mathfrak{g}, M)/\mathrm{Der}_{\mathrm{Inn}}(\mathfrak{g}, M).$$

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My Interests in Lie Algebras

If $\mathcal{H}$ is a complex Hilbert space, then $\mathcal{B}(\mathcal{H})$, the set of bounded linear operators on $\mathcal{H}$ is a vector space over $\mathbb{C}$.

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4. Are there interesting subalgebras?

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