Embeddings of Trees in the Hyperbolic Disk

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Motivation

- Interpolation Problem
  - Can we build a bounded analytic function on the entire disk from knowing the values on a discrete set of points in the disk?
  - Is the function on the disk unique?

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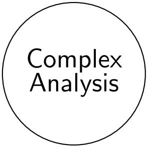
  - Can we build a bounded analytic function on the entire disk from knowing the values on a discrete set of points in the disk?
  - Is the function on the disk unique?
- Possible Areas for Undergraduate Research
  - Discrete structures simplify computations such as integration.

Connections between Branches of Mathematics



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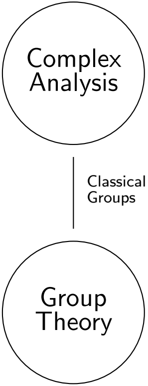

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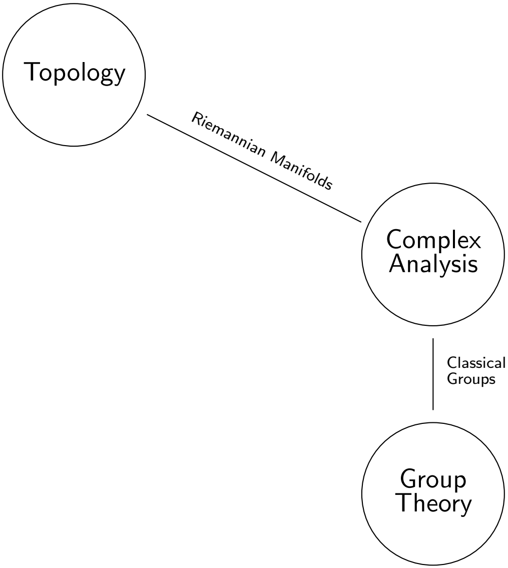

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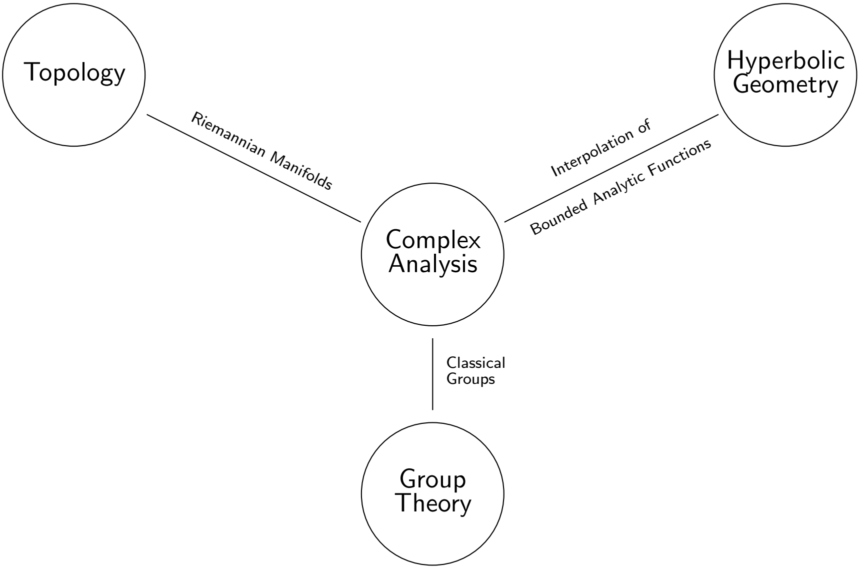

Adgenda

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5. Embeddings of Trees into the Disk
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7. Analysis with Trees
    1. Laplacian Operator on Trees
    2. Bloch Functions on Trees

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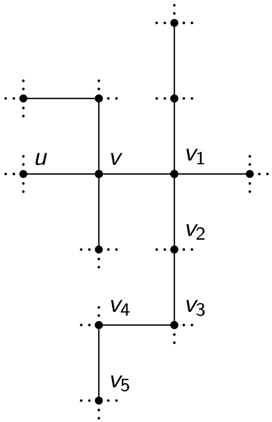
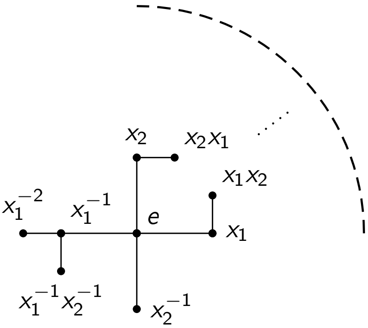
Trees



- A *Tree* is an acyclic, connected graph. A tree is said to be *homogeneous* if every vertex has the same number of edges, denoted by d. The number d is called the degree of the tree.
- Vertices u and v are *associate*, $u \sim v$, if there is an edge connecting them.
- A *path* is a sequence of vertices $[v_1, v_2, v_3, v_4, v_5, \ldots]$ where $v_i \sim v_{i+1}$ and $v_i \neq v_{i+2}$.
- Define $d_T(u, v)$ to be the length of the path connecting u and v. Then d_T is a *metric* on T.
- A tree *isomorphism* is an isometry with respect to d_T.

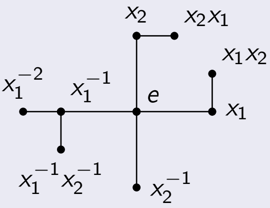
Topological Structure of Trees



- Let $\cong$ be the equivalence relation on the set of infinite paths generated by $[v_0, v_1, \ldots] \cong [v_1, v_2, \ldots]$.
- An end of T is an equivalence class under $\cong$. The set of ends of T is denoted by Ω.
- We can define a topology on Ω for which it is Hausdorff, compact and totally disconnected.
- If we define $T^* = T \cup \Omega$, then T^* is sequentially compact, which implies T^* is compact.

Algebraic Structure of Trees

Trees as Cayley Graphs



- Let $X = \{x_1, x_2\}$ and F_X be the free group with generators X.
- F_X is made up of "words" comprised of the letters $\{x_1, x_2, x_1^{-1}, x_2^{-1}\}$. We call e the *empty word*.
- Since F_X is free, then the *Cayley Graph* is a rooted tree of degree 4 with root e.
- We say that the tree T is generated by $\{x_1, x_2\}$.

NB! The trees discussed in this talk are of even degree. Trees of odd degree can be constructed with some technical changes, but all results will be the same.

Actions on Trees

Translation to the Root

Let $h \in F_X$. We can define the map $\lambda_h : T \to T$ as follows:

$$\lambda_h(v) = hv.$$

This map makes sense when we think of the vertices of T as being elements in F_X. The map λ_h is a left group action of F_X on T. The effect of λ_h is to translate the vertex h to the root of T

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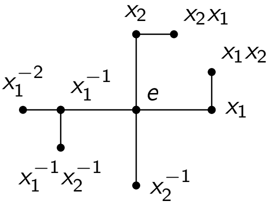

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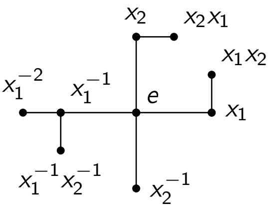

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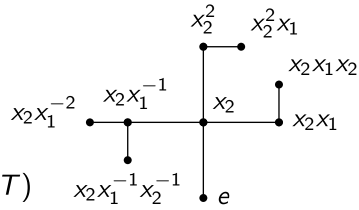
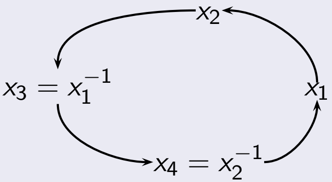


Tree $\lambda_{x_2}(T)$

More Actions on Trees

Rotation about the Root

Consider the tree T generated by $\{x_1, x_2\}$. The set of generators and inverses is thus $\{x_1, x_2, x_1^{-1}, x_2^{-1}\}$. If we think of this as a cycle

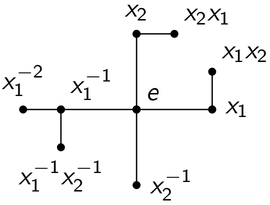


then we can define the map $\psi(x_j) = x_{j+1}$. We can extend the map ψ to a left action on T by defining $\psi(x_i x_j) = x_{i+1} x_{j+1}$.

The map ψ is called the rotation action about the root.

Example of Rotation about the Root

Let us consider the tree T generated by $\{x_1, x_2\}$ as shown below.



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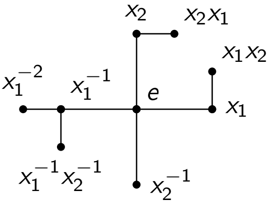
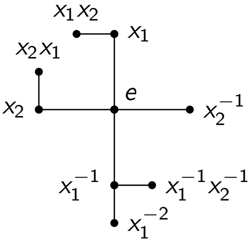
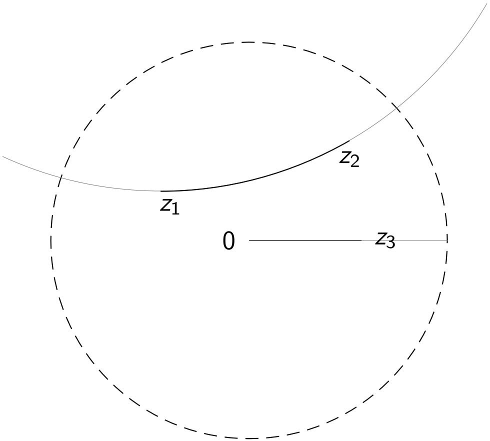


Then applying ψ to T, we have the following rotated tree



Hyperbolic Geometry of the Disk

In order to "fit" an infinite tree inside the unit disk, we must use a geometry other than Euclidean. For this job, the hyperbolic geometry on the disk is ideal.



- The hyperbolic metric on the unit disk is given by

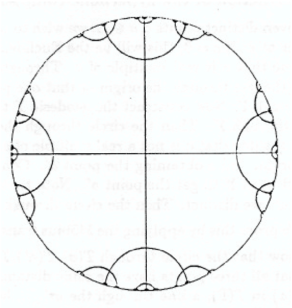
$$\rho(z_1, z_2) = 2\tanh^{-1}\left|\frac{z_1 - z_2}{1 - z_1\overline{z_2}}\right|$$
$$\rho(0, z_2) = 2\tanh^{-1}|z_2|.$$

- The shortest distance between two points z_1 and z_2 is the arc of the circle passing through z_1 and z_2 that is orthogonal to the unit circle.

Embeddings of Trees into the Disk

Let $\Phi : T \to \mathbb{D}$ be a map which sends vertices of T to points in $\mathbb{D}$. This map is an embedding of T into $\mathbb{D}$ if it satisfies the following rules:

- The root of T is mapped to 0.
- The edges of T are mapped to geodesics in $\mathbb{D}$ under the hyperbolic metric.
- Each edge is to have equal (hyperbolic) length, denoted by r.
- No infinite paths intersect excepts at the origin in $\mathbb{D}$.



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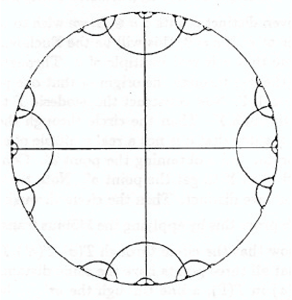

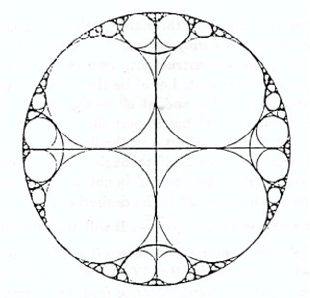

Figures from [2].

Examples of Non-Optimal Embeddings

An embedding Φ is called optimal if it satisfies the additional rule:

- The boundary of the disk must be filled up.

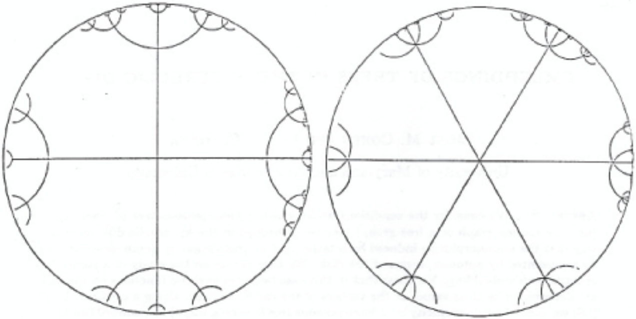

Figure from [2].

The Optimal Embedding

Theorem ([2] Theorem 3)

Let $\Phi : T \to \mathbb{D}$ be a mapping of T which sends neighbors to points in $\mathbb{D}$ of hyperbolic distance r and $r_1 = \cos\left(\frac{\pi}{2t}\right)$. Then

1. *If $0 < r < r_1$, then Φ is not an embedding.*
2. *If $r_1 \leq r < 1$, then Φ is an embedding and the set L of limit points of $\Phi(T)$ is contained in $\partial\mathbb{D}$.*
    1. *If $r_1 < r < 1$, then L is a set of measure 0.*
    2. *If $r = r_1$, then $L = \partial\mathbb{D}$.*

Interpolation Problem

Definition

A sequence $\{z_n\} \subset \mathbb{D}$ is said to be interpolating if for every bounded sequence $\{w_n\} \subset \mathbb{C}$ there exists a function f analytic in $\mathbb{D}$ such that $f(z_n) = w_n$.

In particular, if $w_n = 0$ for all n, does there exist a non-zero analytic function $f : \mathbb{D} \to \mathbb{C}$ such that $f(z_n) = 0$? For this to occur, the Blaschke condition must hold, that is

$$\sum_n (1 - |z_n|) < \infty.$$

If T is a homogeneous tree optimally embedded in $\mathbb{D}$, and the set of vertices is thought of as an interpolating sequence, then the only function for which f is zero on the vertices is the zero function.

Theorem

The restriction function from the set of bounded analytic functions on $\mathbb{D}$ to bounded functions on T is a monomorphism.

Laplacian Operator on Trees

Definition

If $u(x, y)$ is twice-differentiable on some $\Omega \subset \mathbb{R}^2$, then the Laplacian of u is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

The function u is called harmonic if $\Delta u = 0$ on Ω.

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If T is a homogeneous tree of degree d, then a function $f : T \to \mathbb{C}$ is called harmonic if for every vertex v,

$$f(v) = \frac{1}{d} \sum_{w \sim v} f(w).$$

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If we define a Laplacian on T, then it should be 0 on all harmonic functions on T. To do this, we can define Δ on T by

$$\Delta f(v) = \frac{1}{d} \sum_{w \sim v} f(w) - f(v).$$

Bloch Functions on Trees

Definition

A function $f : T \to \mathbb{C}$ is called Bloch if

$$\beta_f = \sup_{v \sim w} |f(v) - f(w)| < \infty.$$

The quantity β_f is called the Bloch constant of f. The set of Bloch functions on a tree T is denoted by $\mathscr{B}(T)$.

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This is analogous to the continuous case on the unit disk.

Analogies Between $\mathscr{B}(T)$ and $\mathscr{B}(\mathbb{D})$

- Let $v_0 \in T$ be a fixed vertex. Then $f : T \to \mathbb{C}$ is Bloch if and only if the family $\{f \circ S - f(S(v_0)) : S \in \text{Aut}\,(T)\}$ is normal.

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  $f : \mathbb{D} \to \mathbb{C}$ is Bloch if and only if the family $\{f \circ S - f(S(0)) : S \in \text{Aut}\,(\mathbb{D})\}$ is normal.

- The little Bloch space $\mathscr{B}_0(T)$ is defined to be the set of functions $f \in \mathscr{B}(T)$ such that for any sequence of pairs of neighboring vertices $\{(u_n, v_n)\}$ approaching the boundary ∂T, we have

$$\lim_{n\to\infty} |f(v_n) - f(u_n)| = 0.$$

$f \in \mathscr{B}_0(\mathbb{D})$ if and only if $\lim\limits_{|z|\to 1^-} (1 - |z|^2)\left|f'(z)\right| = 0.$

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- ${\mathscr{B}_0}^{**} \cong \mathscr{B}$.

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2. If we have a function defined on a tree embedded optimally in the unit disk, we can interpolate to a unique bounded analytic function on the entire unit disk.
3. Harmonic Analysis can be performed on discrete structures such as Trees.
4. Robert can not give a Graduate Seminar talk without discussing the Bloch Space.

References

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