The Spectrum of an Isometric Composition Operator on the Bloch Space in the Polydisk

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Composition Operators...A Philosophic View

Given a Banach space X of analytic functions, we define the composition operator with symbol φ, denoted C_φ, by

$$C_\varphi(f) = f \circ \varphi, \quad f \in X.$$

A goal in studying composition operators is to link the function theory of φ to the operator theory of C_φ.

Typical Questions

- For what φ does C_φ map into X?
- For what φ is C_φ bounded? linear? invertible? compact?
- What is the spectrum of C_φ for various φ?

Research Goal

The goal of our research is to completely describe, on the Bloch space, the symbol φ which induces an isometric composition operator C_φ. Secondly, we wish to describe the spectrum of such operators.

Outline of Talk

Facts from Functional Analysis

Let X and Y be a (complex) Banach spaces and let T be a linear operator on X.

1. We say $T : X \to Y$ is bounded if $\exists C > 0$ such that

$$||Tx||_Y \leq C\,||x||_X\,, \quad \forall x \in X.$$

2. The norm of the operator T is defined as

$$||T|| = \sup_{||x||_X = 1} ||Tx||_Y\,.$$

3. We say $T : X \to X$ is an isometry if $||Tx|| = ||x||$ for all $x \in X$.
4. The spectrum of T, denoted $\sigma(T)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

Basic Facts:

- An isometry is necessarily injective.
- For a bounded operator T, T is invertible if and only if T is bounded below and has dense range.
- If T is an invertible bounded linear operator, then the inverse is bounded and linear (Inverse Mapping Theorem).
- The operator norm of an isometry is 1. However, there are operators with norm 1 that are not isometries.
- For an operator T, $\sigma(T) \subseteq \{|z| \leq ||T||\}$.
- The spectrum of an operator is a non-empty compact subset of $\mathbb{C}$.
- The spectrum of an isometry is a non-empty compact subset of $\overline{\mathbb{D}}$.

Our Starting Point...The Bloch Space in $\mathbb{D}$

An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be Bloch provided

$$\begin{aligned}\beta_f &= \sup_{z\in\mathbb{D}}(1-|z|^2)\left|f'(z)\right| \\ &= \sup_{z\in\mathbb{D}} \sup_{u\in\mathbb{C}\setminus\{0\}} \frac{|f'(z)u|}{\frac{|u|}{1-|z|^2}} < \infty.\end{aligned}$$

The Bloch space defined as $\mathcal{B} = \{f \in H(\mathbb{D}) : \beta_f < \infty\}$ is a Banach space of analytic functions with norm

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f.$$

Goal

In this talk, we will completely describe the spectrum of isometric composition operators on the Bloch space of $\mathbb{D}$. We then see to what extent these results generalize to the Bloch space of the unit polydisk $\mathbb{D}^n$.

An Old Idea Finds the Treasure Map

Exercise (Conway, 1990)

If X is a Banach space and $T : X \to X$ is an isometry, then either $\sigma(T) \subseteq \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$.

This result is very general. It applies to any complex Banach space and any isometry, not necessarily a composition operator.

Lemma (A. & Colonna)

Let X be a Banach space and $T : X \to X$ be an isometry. If T is invertible then $\sigma(T) \subseteq \partial\mathbb{D}$. If T is not invertible then $\sigma(T) = \overline{\mathbb{D}}$.

A New Idea Finds the Treasure

Theorem (Colonna, 2005)

The symbols φ of the isometric composition operator on the Bloch space of the unit disk are precisely

1. *functions of the form $\varphi(z) = \lambda z$ for $|\lambda| = 1$, and*
2. *functions of the form $\varphi = gB$ where g is a nonvanishing analytic function from $\mathbb{D} \to \overline{\mathbb{D}}$, and B is a Blaschke product whose zero set Z contains 0 and an infinite sequence $\{z_n\}$ such that $|g(z_n)| \to 1$ and*

$$\lim_{n\to\infty} \prod_{\zeta\in Z, \zeta\neq z_n} \left| \frac{z_n - \zeta}{1 - \overline{z_n}\zeta} \right| = 1.$$

We now can connect the functional analysis of the symbol to the spectrum of the induced composition operator.

The Treasure (in One Dimension)

Theorem (A. & Colonna)

Let φ be the symbol of an isometric composition operator on the Bloch space for the unit disk.

1. *If φ is not a rotation, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*
2. *If φ is a rotation with rotation angle $\theta \notin \pi\mathbb{Q}$, then $\sigma(C_\varphi) = \partial\mathbb{D}$.*
3. *If φ is a rotation with rotation angle $\theta \in [0, 2\pi) \cap \pi\mathbb{Q}$, then $\sigma(C_\varphi)$ is the set of m^{th} roots of unity, where m is the smallest positive integer such that $m\theta = 2\pi$.*

We have a complete characterization of the spectrum of isometric composition operators on the Bloch space in $\mathbb{D}$. Our next goal is to see what results can be generalized to higher dimensions.

Holomorphic Functions in Higher Dimensions

Goal

To move to higher dimensions, we must first generalize the notion of analytic functions.

Analytic functions (in one variable) can be characterized in several different ways, i.e. having a power series representation or satisfying the Cauchy-Riemann equations. We will use this notion to generalize to higher dimensions.

Let $\Omega \subset \mathbb{C}^n$ be open. Then a function $f : \Omega \to \mathbb{C}$ is called holomorphic if f is analytic in each variable separately. We denote the space of holomorphic functions on Ω by $H(\Omega)$.

Let $\Omega_1 \subset \mathbb{C}^k$ and $\Omega_2 \subset \mathbb{C}^m$ be open. Then $f : \Omega_1 \to \Omega_2$ is called a holomorphic map if $f = (f_1, \ldots, f_m)$ for $f_1, \ldots, f_m \in H(\Omega_1)$.

Extending the Unit Disk into Higher Dimensions

The Uniformization theorm states there are only three types of simply-connected open subsets of a Riemann surface: $\widehat{\mathbb{C}}$, $\mathbb{C}$, and $\mathbb{D}$. So, there are not many choices of domains on which to study.

In several complex variables, there is no notion of the Uniformization theorem. When generalizing to higher complex dimensions, there are many choices as to the domain to study. There are two very well-studied domains:

The Unit Ball

$$\mathbb{B}_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : ||z|| = \left(\sum_{j=1}^{n} |z_j|^2\right)^{1/2} < 1\}.$$

The Unit Polydisk

$$\mathbb{D}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1 \text{ for each } 1 \leq i \leq n\}.$$

The Bloch Seminorm in Higher Dimensions

Recall the Bloch semi-norm in one dimension:

$$\beta_f = \sup_{z\in\mathbb{D}} \sup_{u\in\mathbb{C}\setminus\{0\}} \frac{|f'(z)u|}{\frac{|u|}{1-|z|^2}}.$$

In higher dimensions, the numerator can be generalized by a directional derivative.

$$f'(z)u \rightsquigarrow \langle \nabla f(z), \overline{u} \rangle = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) u_j.$$

The denominator can be generalized by the function

$$H_z(u, \overline{v}) = \sum_{j=1}^{n} \frac{u_j \overline{v_j}}{(1-|z_j|^2)^2}, \quad u, v \in \mathbb{C}^n$$

called the Bergman metric on $\mathbb{D}^n$.

Bloch Functions in the Polydisk

A holomorphic function $f : \mathbb{D}^n \to \mathbb{C}$ is said to be Bloch if

$$\beta_f = \sup_{z \in \mathbb{D}^n} Q_f(z) < \infty$$

where

1. $Q_f(z) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \dfrac{|\langle \nabla f(z), \overline{u} \rangle|}{H_z(u, \overline{u})^{1/2}}$
2. $\langle \nabla f(z), \overline{u} \rangle = \sum_{k=1}^{n} \dfrac{\partial f}{\partial z_k}(z) u_k$
3. H_z is the Bergman Metric on $\mathbb{D}^n$.

The space $\mathcal{B}(\mathbb{D}^n)$ of all Bloch functions on $\mathbb{D}^n$ is a Banach space under the norm

$$||f||_{\mathcal{B}} = \beta_f + |f(0)|.$$

What Happens when $n = 1$?

The first thing we want to do is make sure that this definition agrees with our definition of Bloch on $\mathbb{D}$. For $n = 1$, we have

1. $$H_z(u,\overline{u})^{1/2} = \left(\sum_{k=1}^{1} \frac{u_k\overline{u_k}}{(1-|z_k|^2)^2}\right)^{1/2} = \frac{|u|}{1-|z|^2}.$$
2. $$|\langle \nabla f(z), \overline{u}\rangle| = \left|\sum_{k=1}^{1} \frac{\partial f}{\partial z_k}(z)u_k\right| = \left|f'(z)u\right|.$$
3. $$Q_f(z) = \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|(\nabla f)(z)u|}{H_z(u,\overline{u})^{1/2}} = (1-|z|^2)\left|f'(z)\right|.$$

The Induced Bloch Semi-norm on $\mathbb{D}$

$$\beta_f = \sup_{z\in\mathbb{D}^1} Q_f(z) = \sup_{z\in\mathbb{D}}(1-|z|^2)\left|f'(z)\right|.$$

Isometric Composition Operators on $\mathcal{B}(\mathbb{D}^n)$

Theorem

Let $\varphi : \mathbb{D}^n \to \mathbb{D}^n$ be a holomorphic map. Then C_φ is a bounded operator from $\mathcal{B}(\mathbb{D}^n)$ into itself.

One goal of our research is to completely classify the isometric composition operators on $\mathcal{B}(\mathbb{D}^n)$ based on properties of the symbol. We are searching for a result of the form

The holomorphic map $\varphi : \mathbb{D}^n \to \mathbb{D}^n$ has properties $\mathcal{P}(\varphi)$ if and only if C_φ is an isometry on $\mathcal{B}(\mathbb{D}^n)$.

Theorem (Cohen & Colonna)

Let $\varphi(z) = (h_1(z_1), h_2(z_2), \ldots, h_n(z_n))$ where $h_1, \ldots, h_n \in H(\mathbb{D}, \mathbb{D})$ such that $h_j(0) = 0$ and $\beta_{h_j} = 1$ for all $j = 1 \ldots n$. Then C_φ is an isometry on $\mathcal{B}(\mathbb{D}^n)$.

The Spectrum of an Isometric Composition Operator

Let $\varphi : \mathbb{D}^n \to \mathbb{D}^n$ be defined by $\varphi(z) = (\lambda_1 z_1, \ldots, \lambda_n z_n)$ where $|\lambda_j| = 1$ for $j = 1, \ldots, n$. The induced composition operator C_φ is an isometry.

It is clear that $\varphi^{-1}(z) = (\lambda_1^{-1} z_1, \ldots, \lambda_n^{-1} z_n)$. This induces the composition operator $C_{\varphi^{-1}} = C_\varphi^{-1}$.

Since C_φ is invertible, $0 \notin \sigma(C_\varphi)$. So, $\sigma(C_\varphi) \subseteq \partial\mathbb{D}$.

Narrowing Down on the Spectrum

We now want to determine what subset of $\partial\mathbb{D}$ comprises the spectrum of C_φ.

For all $k_1, \ldots, k_n \in \mathbb{Z}_+ \cup \{0\}$, the function $f(z) = z_1^{k_1} \cdots z_n^{k_n}$ is a Bloch function. Also, notice that

$$C_\varphi(f)(z) = f(\varphi(z)) = f(\lambda_1 z_1, \ldots, \lambda_n z_n) = \lambda_1^{k_1} \cdots \lambda_n^{k_n} f(z).$$

So f is an eigenfunction of C_φ with eigenvalue $\lambda_1^{k_1} \cdots \lambda_n^{k_n}$. If we define

$$Q_\lambda := \{\lambda_1^{k_1} \cdots \lambda_n^{k_n} : k_1, \ldots, k_n \geq 0\}$$

then

$$\overline{Q_\lambda} \subseteq \sigma(C_\varphi).$$

Exploring the Set Q_λ

Case 1.

Suppose $\arg(\lambda_j)$ is not a rational multiple of π for some $j = 1, \ldots, n$.

Then Q_λ is dense in $\partial\mathbb{D}$. So $\overline{Q_\lambda} = \partial\mathbb{D}$. Thus $\sigma(C_\varphi) = \partial\mathbb{D}$.

Case 2.

Suppose $\arg(\lambda_j)$ is a rational multiple of π for all $j = 1, \ldots, n$. Let $m_1, \ldots, m_n$ be the orders of $\lambda_1, \ldots, \lambda_n$ respectively.

The group Q_λ generated by $\{\lambda_1, \ldots, \lambda_n\}$ is of finite order equal to $\mathrm{lcm}(m_1, \ldots, m_n)$.

The Spectrum in the Finite Order Case

Let $\mu \in \partial\mathbb{D} \setminus Q_\lambda$. We want to show $C_\varphi - \mu I$ is invertible. Since $C_\varphi - \mu I$ is bounded, it suffices to show $C_\varphi - \mu I$ is bijective. This is equivalent to showing for all $g \in \mathcal{B}(\mathbb{D}^n)$ there exists a unique $f \in \mathcal{B}(\mathbb{D}^n)$ such that $C_\varphi(f) - \mu f = g$.

Let $\alpha = \mathrm{lcm}(m_1, \ldots, m_n)$. Then $\varphi^\alpha = \mathrm{id}$. We can now formulate the invertibility of $C_\varphi - \mu I$ in terms of a system of equations.

So $C_\varphi - \mu I$ is invertible if and only for every $g \in \mathcal{B}(\mathbb{D}^n)$ the following system has a unique solution $f \in \mathcal{B}(\mathbb{D}^n)$.

$$\begin{array}{lcllcl}
f(\varphi(z)) & - & \mu f(z) & = & g(z) \\
f(\varphi^2(z)) & - & \mu f(\varphi(z)) & = & g(\varphi(z)) \\
 & \vdots & & \vdots & \\
f(z) & - & \mu f(\varphi^{\alpha-1}(z)) & = & g(\varphi^{\alpha-1}(z)).
\end{array}$$

So, the system of equations has a unique solution if and only if the matrix A in the following system is non-singular:

$$\begin{bmatrix} -\mu & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & & & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & -\mu \end{bmatrix} \begin{bmatrix} f(z) \\ f(\varphi(z)) \\ \vdots \\ \vdots \\ f(\varphi^{\alpha-2}(z)) \\ f(\varphi^{\alpha-1}(z)) \end{bmatrix} = \begin{bmatrix} g(z) \\ g(\varphi(z)) \\ \vdots \\ \vdots \\ g(\varphi^{\alpha-2}(z)) \\ g(\varphi^{\alpha-1}(z)) \end{bmatrix}.$$

The determinant of A is easily calculated as $|A| = (-1)^{\alpha}(\mu^{\alpha} - 1)$. This is clearly zero if and only if $\mu^{\alpha} = 1$ which implies that $\mu \in Q_{\lambda}$, against the assumption.

Thus, for every $g \in \mathcal{B}(\mathbb{D}^n)$, there exists $f(z) = \sum_{j=1}^{\alpha} a_j g(\varphi^{j-1}(z))$ such that $C_{\varphi}(f) - \mu f = g$. Since each $g \circ \varphi^k$ is Bloch for all $k = 0, \ldots, \alpha - 1$, f is Bloch also.

Spectrum Result (So Far)

Proposition (A. & Colonna)

For $z \in \mathbb{D}^n$, let $\varphi(z) = (\lambda_1 z_1, \ldots, \lambda_n z_n)$ with $|\lambda_j| = 1$ for $j = 1, \ldots, n$, and C_φ be the induced composition operator on $\mathbb{B}(\mathbb{D}^n)$.

1. *If $\arg(\lambda_j)$ has infinite order for some $j = 1, \ldots, n$, then $\sigma(C_\varphi) = \partial\mathbb{D}$.*
2. *If $\arg(\lambda_j)$ has finite order for all $j = 1, \ldots, n$, then $\sigma(C_\varphi)$ is the finite group generated by $\{\lambda_1, \ldots, \lambda_n\}$.*

Further Direction and Open Questions

1. If φ has a component which is not a rotation, but the other type of isometry on $\mathbb{D}$, is $\sigma(C_\varphi) = \overline{\mathbb{D}}$?
2. When a composition operator on the Bloch space is invertible, is the inverse a composition operator?
3. Are there other holomorphic maps φ which induce an isometric composition operator on the Bloch space of the polydisk besides the ones whose components are defined in terms of isometries on $\mathbb{D}$?
4. A positive answer to question 2 and a negative answer to question 3 will yield a complete classification of the spectrum in the case of the polydisk.

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