On the Spectrum of an Isometric Composition Operator on the Bloch Space in the Polydisk

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Research Goals

The goal of our research is to completely classify the symbols φ which induce isometric composition operators C_φ on the Bloch space of a bounded symmetric domain D. To this, we also look to classify the spectrum of such isometric composition operators.

Purpose of this Talk

I will discuss our classification of the spectrum of isometric composition operators on the Bloch space of the unit polydisk $\mathbb{D}^n$. I will also show how this classification implies the complete classification in the case of the unit disk.

Agenda

Factual Round-Up

Definition

Let X be a complex Banach space. A linear operator $T : X \to X$ is called an isometry if $||Tx|| = ||x||$ for all $x \in X$.

The spectrum, point spectrum and approximate point spectrum of a bounded linear operator T on X are defined respectively as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$
$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not 1-1}\},$$
$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is 1-1 but not bounded below}\}.$$

Proposition (Conway,1990)

If T is a bounded linear operator on a complex Banach space X, then $\partial\sigma(T) \subseteq \sigma_{ap}(T)$.

The following is found as an exercise in A Course in Functional Analysis, John Conway, 1990.

Exercise

If X is a Banach space and $T : X \to X$ is an isometry, then either $\sigma(T) \subseteq \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$.

Sketch of Proof.

$\sigma(T) \subseteq \overline{\mathbb{D}}$ since $||T|| = 1$.

Case 1: Suppose T is invertible.

- $0 \notin \sigma(T)$ and $z \mapsto z^{-1}$ is analytic in some $G \subset \sigma(T)$.
- $\sigma(T^{-1}) = \sigma(T)^{-1} = \{\lambda^{-1} \in \mathbb{C} : \lambda \in \sigma(T)\} \subseteq \overline{\mathbb{D}}$.

Case 2: Suppose T is not invertible.

- $0 \in \sigma(T)$.
- Since $\partial\sigma(T) \subseteq \sigma_{ap}(T) \subseteq \partial\mathbb{D}$ it follows that $\sigma(T) = \overline{\mathbb{D}}$.

Complete Classification for the Unit Disk

Theorem (Colonna, 2005)

The symbols φ of the isometric composition operator on the Bloch space of the unit disk are precisely

1. *functions of the form $\varphi(z) = \lambda z$ for $|\lambda| = 1$, and*
2. *functions of the form $\varphi = gB$ where g is a nonvanishing analytic function from $\mathbb{D}$ into $\overline{\mathbb{D}}$, and B is a Blaschke product whose zero set Z contains 0 and an infinite sequence $\{z_n\}$ such that $|g(z_n)| \to 1$ and*

$$\lim_{n\to\infty} \prod_{\zeta\in Z, \zeta\neq z_n} \left| \frac{z_n - \zeta}{1 - \overline{z_n}\zeta} \right| = 1.$$

As a consequence of the previous Exercise and Theorem, we have a complete classification of the spectrum for the case of the unit disk.

Theorem (A. & Colonna)

Let φ be the symbol of an isometric composition operator on the Bloch space of the unit disk.

1. *If φ is not a rotation, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*
2. *If φ is a rotation with rotation angle $\theta \notin \pi\mathbb{Q}$, then $\sigma(C_\varphi) = \partial\mathbb{D}$.*
3. *If φ is a rotation with rotation angle $\theta \in \pi\mathbb{Q}$, then $\sigma(C_\varphi)$ is the set of m^{th} roots of unity, where m is the order of $e^{i\theta}$.*

Bloch Space of the Polydisk

As we have seen, the Bloch space for a bounded symmetric domain D is a Banach space defined by

$$\mathcal{B}(D) = \{f \in H(D) : \beta_f < \infty\}$$

where

$$\beta_f = \sup_{z \in D} \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{|(\nabla f)(z)u|}{H_z(u, \overline{u})^{1/2}}.$$

If D is taken to be the unit polydisk $\mathbb{D}^n$, then the Bergman metric becomes

$$H_z(u, \overline{v}) = \sum_{k=1}^{n} \frac{u_k \overline{v_k}}{(1 - |z_k|^2)^2}.$$

Isometric Composition Operators

We have seen in the previous talk that any holomorphic map $\varphi : \mathbb{D}^n \to \mathbb{D}^n$ induces a bounded composition operator C_φ on $\mathcal{B}(\mathbb{D}^n)$.

Theorem (Cohen & Colonna)

Let $\varphi(z) = (h_1(z_1), h_2(z_2), \ldots, h_n(z_n))$ where $h_j \in H(\mathbb{D}, \mathbb{D})$ such that $h_j(0) = 0$ and $\beta_{h_j} = 1$ for all $j = 1, \ldots, n$. Then C_φ is an isometry on $\mathcal{B}(\mathbb{D}^n)$.

We will consider the spectrum of the composition operators induced by such symbols.

Symbols with Non-Rotation Component

Let $\varphi = (h_1(z_1), \dots, h_n(z_n))$ where $h_j \in H(\mathbb{D}, \mathbb{D})$ such that $h_j(0) = 0$ and $\beta_{h_j} = 1$ for all $j = 1, \dots, n$. Suppose that h_j is not a rotation for some $j = 1, \dots, n$.

Suppose C_φ is surjective and let $1 \leq j \leq n$ be the component defined as $h_j(z_j) = g(z_j)B(z_j)$ with $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$ the zeros of the Blaschke product B.

Since $\pi_j(z) = z_j$ is Bloch, there exists a Bloch function f such that $f \circ \varphi = \pi_j$. Thus for any $k \in \mathbb{N}$, and $z_j = 0$, for $j \neq k$,

$$a_k = f(\varphi(z_1, \dots, z_{j-1}, a_k, z_j, \dots, z_n) = f(0).$$

Since the set of such a_k is infinite, we obtain a contradiction. Therefore, C_φ is not invertible and $\sigma(C_\varphi) = \overline{\mathbb{D}}$.

Symbols with Rotation Components

Recall that the automorphisms of $\mathbb{D}^n$ that fix 0 are of the form $(z_1, \ldots, z_n) \mapsto (\lambda_1 z_{\tau(1)}, \ldots, \lambda_n z_{\tau(n)})$, with $|\lambda_j| = 1, j = 1, \ldots, n$, and $\tau \in S_n$.

Let $\varphi : \mathbb{D}^n \to \mathbb{D}^n$ be defined by $\varphi(z) = (\lambda_1 z_{\tau(1)}, \ldots, \lambda_n z_{\tau(n)})$ where $|\lambda_j| = 1$ for $j = 1, \ldots, n$ and $\tau \in S_n$.

The induced C_φ is invertible. So, $\sigma(C_\varphi) \subseteq \partial\mathbb{D}$.

For all $k_1, \ldots, k_n \in \mathbb{Z}_+ \cup \{0\}$, the function $f(z) = z_{\tau(1)}^{k_1} \cdots z_{\tau(n)}^{k_n}$ is Bloch and is an eigenfunction of C_φ with eigenvalue $\lambda_1^{k_1} \cdots \lambda_n^{k_n}$.

Consider the group G generated by $\lambda_1, \ldots, \lambda_n$ and the m_τ^{th} roots of unity for $m_\tau = \text{ord}(\tau)$. Then

$$\overline{G} \subseteq \sigma(C_\varphi).$$

Exploring the Set G

Case 1: Infinite Order

Suppose $\arg(\lambda_j)$ is not a rational multiple of π for some $j = 1, \ldots, n$.

Then G is dense in $\partial\mathbb{D}$. So $\overline{G} = \partial\mathbb{D}$. Thus $\sigma(C_\varphi) = \partial\mathbb{D}$.

Case 2: Finite Order

Suppose $\arg(\lambda_j)$ is a rational multiple of π for all $j = 1, \ldots, n$. Let $m_1, \ldots, m_n$ be the orders of $\lambda_1, \ldots, \lambda_n$ respectively.

The group G has finite order $\operatorname{lcm}(m_1, \ldots, m_n, m_\tau)$.

The Spectrum in the Finite Order Case

We now show that for $\mu \in \partial\mathbb{D}$, $C_\varphi - \mu I$ is invertible if and only if $\mu \notin G$. The invertibility of $C_\varphi - \mu I$ is equivalent to showing that for all $g \in \mathcal{B}(\mathbb{D}^n)$ there exists a unique $f \in \mathcal{B}(\mathbb{D}^n)$ such that $C_\varphi(f) - \mu f = g$.

Let $\alpha = \text{lcm}(m_1, \ldots, m_n, m_\tau)$. So $C_\varphi - \mu I$ is invertible if and only if for every $g \in \mathcal{B}(\mathbb{D}^n)$ the following system has a unique solution $f \in \mathcal{B}(\mathbb{D}^n)$:

$$\begin{array}{rclcl} f(\varphi(z)) & - & \mu f(z) & = & g(z) \\ f(\varphi^2(z)) & - & \mu f(\varphi(z)) & = & g(\varphi(z)) \\ & \vdots & & \vdots & \\ f(z) & - & \mu f(\varphi^{\alpha-1}(z)) & = & g(\varphi^{\alpha-1}(z)). \end{array}$$

So, the system of equations has a unique solution if and only if the matrix A in the following system is non-singular:

$$\begin{bmatrix} -\mu & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & & & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & -\mu \end{bmatrix} \begin{bmatrix} f(z) \\ f(\varphi(z)) \\ \vdots \\ \vdots \\ f(\varphi^{\alpha-2}(z)) \\ f(\varphi^{\alpha-1}(z)) \end{bmatrix} = \begin{bmatrix} g(z) \\ g(\varphi(z)) \\ \vdots \\ \vdots \\ g(\varphi^{\alpha-2}(z)) \\ g(\varphi^{\alpha-1}(z)) \end{bmatrix}.$$

The determinant of A is easily calculated as $|A| = (-1)^{\alpha}(\mu^{\alpha} - 1)$. Thus the solution exists and is unique if and only if $\mu^{\alpha} \neq 1$, i.e. $\mu \notin G$.

Thus, for every $g \in \mathcal{B}(\mathbb{D}^n)$, the unique solution $f(z) = \sum_{j=1}^{\alpha} a_j g(\varphi^{j-1}(z))$, such that $C_{\varphi}(f) - \mu f = g$, is Bloch since each $g \circ \varphi^k$ is Bloch.

Classification of Isometric Composition Operators

Our current classification can be summarized as follows:

Proposition (A. & Colonna)

Let $\varphi(z) = (h_1(z_1), \ldots, h_n(z_n))$ be a holomorphic map such that $h_j(0) = 0$ and $\beta_{h_j} = 1$ for all $j = 1, \ldots, n$, and C_φ be the induced composition operator on $\mathcal{B}(\mathbb{D}^n)$.

1. *If h_j is not a rotation of the identity for some $j = 1, \ldots, n$, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*
2. *If $h_j(z_j) = \lambda_j z_{\tau(j)}$ for all $j = 1, \ldots, n$ and some $\tau \in S_n$ such that λ_k has infinite order for some $k = 1, \ldots, n$, then $\sigma(C_\varphi) = \partial\mathbb{D}$.*
3. *If $h_j(z_j) = \lambda_j z_{\tau(j)}$ for all $j = 1, \ldots, n$ and some $\tau \in S_n$ such that λ_k has finite order for all $k = 1, \ldots, n$, then $\sigma(C_\varphi)$ is the finite group generated by $\{\lambda_1, \ldots, \lambda_n, \zeta\}$ where ζ is a primitive m_τ^{th} root of unity, $m_\tau = \mathrm{ord}(\tau)$.*

Further Direction and Open Questions

1. Are there other holomorphic maps φ which induce an isometric composition operator on the Bloch space of the polydisk besides the ones whose components are defined in terms of isometries on $\mathbb{D}$?

2. It can be easily shown that if C_φ is an isometry and φ is not 1-1, then $0 \in \sigma(C_\varphi)$, so that $\sigma(C_\varphi) = \overline{\mathbb{D}}$. Are there any isometric composition operators whose symbols are not onto?

   A negative answer to this question would give us a complete classification of the spectrum.

3. When a composition operator on the Bloch space is invertible, is the inverse a composition operator?

4. What is the classification of the spectrum on $\mathcal{B}(\mathbb{B}_n)$ for $n > 1$? on a general bounded symmetric domain?

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