Introduction to nonlinear FEM

Most physical problems are inherently nonlinear in nature. Such non-linearities arise due to a non-linear strain-displacement (Geometric) relation or due to a non-linear stress-strain (Material) relationship. Geometric nonlinearities occur due to large displacements, large strains, large rotations and so on. Material non-linearities occur when the stress-strain (Constitutive Law) or force-displacement relationship is not linear, or when material properties change with applied loads. For example, the young's modulus parameter may vary with displacement i.e. $E = E(u)$.

Let us consider the axial deformation of a bar with distributed load $q(x)$ as before. This time let us assume a non linear stress-strain relationship of the form

$$\sigma(x) = E(u)\,\varepsilon(x)$$

$$= E(u)\,\frac{du}{dx}$$

Subtituting this into the governing differential equation for the bar is

$$A\,\frac{d\sigma}{dx} + q(x) = 0$$

$$\Rightarrow \quad A\,\frac{d}{dx}\left[E(u)\,\frac{du}{dx}\right] + q(x) = 0$$

To obtain the weak-variational form, we multiply this equation by a test function $v(x)$ over the element and integrate by parts. This yields

$$\int_0^L A \frac{d}{dx}\left[E(u)\frac{du}{dx}\right] v(x)\,dx = -\int_0^L q(x)\,v(x)\,dx$$

$$\Rightarrow \quad A\,E(u)\frac{du}{dx}\,v(x)\Big|_0^L - \int_0^L A\,E(u)\frac{du}{dx}\frac{dv}{dx}\,dx = -\int_0^L q(x)\,v(x)\,dx$$

Applying the Boundary Conditions used as in the linear case, namely,

$$u(0) = 0$$

$$AE(u)\frac{du}{dx}(L) = R \;.$$

where R was a concentrated load at $x = L$, we get,

$$\int_0^L A\,E(u)\frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^L q(x)\,v(x)\,dx + R\,v(L)$$

FINITE ELEMENT DISCRETIZATION:

As before we consider a finite dimensional approximation $u_h(x)$ of the continuous solution $u(x)$ such that; our problem becomes: Find $u_h(x) \in V_N$

$$\int_0^L AE(u_h) \frac{du_h}{dx} \frac{dv}{dx} \, dx = \int_0^L q(x) \, v(x) \, dx + R \, v(L)$$

for all $v(x) \in V_N$ (= finite dimensional space).

Supposing that $V_N = \text{span}\{\phi_1(x), \phi_2(x) \cdots \phi_N(x)\}$ where $\phi_i(x)$ is a basis function for V_N, then we can write, $u_h(x) = \sum_{i=1}^{N} c_i \phi_i(x)$ and $v(x) = \phi_j(x)$ for $j = 1..N$. Using this our discrete variational form becomes:

$$\int_0^L AE\left(\sum_{i=1}^{N} c_i \phi_i\right) \frac{d}{dx}\left(\sum_{i=1}^{N} c_i \phi_i\right) \frac{d\phi_j}{dx} \, dx = \int_0^L q(x) \, \phi_j(x) \, dx + R \, \phi_j(L)$$

$$j = 1..N$$

This can be rewritten as:

$$\sum_{i=1}^{N} c_i \int_0^L AE(u_h) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} = \int_0^L q(x)\, \phi_j(x)\, dx + R\phi_j(L)$$

$$j = 1, 2 \cdots N$$

This can be written in matrix form as

$$K(\vec{c})\,\vec{c} = \vec{F} \qquad \text{— } (*)$$

where

$$K_{ij} = A \int_0^L E\left(\sum_{i=1}^{N} c_i \phi_i(x)\right) \frac{d\phi_j}{dx} \frac{d\phi_i}{dx}\, dx$$

$$F_j = \int_0^L q(x)\, \phi_j(x) dx + R\, \phi_j(L)$$

Notice that solving (*) requires us to solve a nonlinear system of equations and therefore cannot be solved by direct methods.

Let us assume $\boxed{E(u) = E_0\left(1 - \frac{du}{dx}\right)}$

In terms of the coefficients we have,

$$E(u_h) = E_0\left(1 - \frac{d}{dx}u_h\right)$$

$$= E_0\left(1 - \sum_{k=} C_k \frac{d\phi_k}{dx}\right)$$

Then the entries of the stiffness matrix becomes

$$K_{ij} = AE_0 \int_0^L \left(1 - \sum_k C_k \frac{d\phi_k}{dx}\right) \frac{d\phi_j}{dx} \frac{d\phi_i}{dx}\, dx$$

As in the linear case we will express the entries of the global stiffness matrix in terms of local stiffness matrices. For this we partition the domain $[0, L]$ into M subintervals and then we can express $K_{ij} = \sum_{e=1}^{M} K_{ij}^{(e)}$

where

$$K_{ij}^{(e)} = AE_0 \int_{x_e}^{x_{e+1}} \left(1 - \sum_k C_k^e \frac{d\phi_k^e}{dx}\right) \frac{d\phi_j^e}{dx} \frac{d\phi_i^e}{dx}\, dx$$

Each of these local elements $[x_e, x_{e+1}]$ are mapped using linear transformations to $[-1, 1]$ as in the linear case to perform the integral computation. The local basis functions over this reference element $[-1, 1]$ is given by

$$N_1(\xi) = \frac{1-\xi}{2} \qquad N_2(\xi) = \frac{1+\xi}{2}$$

Let us now consider the simple case of the axial bar divided into two elements (i.e. with three nodes x_1, x_2, x_3). We can then write the element stiffness matrix over $[x_e, x_{e+1}]$ $e=1,2$ as:

$$K_{ij}^{(e)} = AE_0 \int_{x_e}^{x_{e+1}} \left(1 - c_1^e \frac{d\phi_1^e}{dx} - c_2^e \frac{d\phi_2^e}{dx}\right) \frac{d\phi_j^e}{dx} \frac{d\phi_i^e}{dx}\, dx$$

which can be computed using the transformation between $[x_e, x_{e+1}]$ and $[-1, 1]$ as before.

The local stiffness matrices work out to

$$K^{(e)} = \frac{AE_0}{h_e^2}\left(h + c_1^e - c_2^e\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where c_1^e, c_2^e correspond to unknown coefficients over each element. For the simple example of two elements $c_1^1 = c_1$, $c_2^1 = c_2$, $c_1^2 = c_2$, $c_2^2 = c_3$.

Next step is to assemble the local stiffness matrices to yield the following global stiffness matrix (for uniform grid with $h_e = h$):

$$K(\vec{c}) = \frac{AE_0}{h^2}\begin{bmatrix} h + c_1 - c_2 & -(h + c_1 - c_2) & 0 \\ -(h + c_1 - c_2) & (2h + c_1 - c_3) & -(h + c_2 - c_3) \\ 0 & -(h + c_2 - c_3) & (h + c_2 - c_3) \end{bmatrix}$$

Note that after assembly one must reduce (or condense) the global stiffness matrix to account for the boundary condition at $x=0$ which yields the following reduced stiffness matrix

$$\tilde{K}(\vec{c}) = \frac{AE_0}{h^2}\begin{bmatrix} 2h-c_3 & -(h+c_2-c_3) \\ -(h+c_2-c_3) & h+c_2-c_3 \end{bmatrix}$$

It must be noted that one must perform similar computations on the right hand side as in the linear case. Suppose we consider $q(x)=0$ (for simplicity) then we obtain the following reduced nonlinear system:

$$\frac{AE_0}{h^2}\begin{bmatrix} 2h-c_3 & -(h+c_2-c_3) \\ -(h+c_2-c_3) & h+c_2-c_3 \end{bmatrix}\begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

or $\quad \tilde{K}(\vec{c})\,\vec{c} = \tilde{\vec{F}}$

To solve this we consider the residual system

$$\vec{\tilde{R}}(\vec{c}) = \tilde{K}(\vec{c})\,\vec{c} - \vec{\tilde{F}} = 0$$

and employ the Newton-Raphson method.

Starting with an initial guess $\vec{C}_0 = \begin{bmatrix} c_2 \\ c_3 \end{bmatrix}_0$

the Newton Raphson iterations yield:

$$\vec{C}_{n+1} = \vec{C}_n - T^{-1}(\vec{C}_n)\,\vec{\tilde{R}}(\vec{C}_n)$$

where $T(\vec{C}_n)$ is the tangent stiffness matrix

given by:

$$T(\vec{C}_n) := \frac{\partial \vec{\tilde{R}}}{\partial \vec{C}_n}(\vec{C}_n)$$

which can be computed as:

$$T(\vec{C}_n) = \frac{A C_0}{h^2} \begin{bmatrix} 2(h - c_3) & -h - 2(c_2 - c_3) \\ -h - 2(c_2 - c_3) & h + 2(c_2 - c_3) \end{bmatrix}_n$$

The inverse of $T(C_n)$ can be computed to be:

$$T^{-1}(C_n) = \frac{h^2}{AE_0(h-2C_2)} \begin{bmatrix} 1 & 1 \\ 1 & \dfrac{2h-2C_3}{h+2C_2-2C_3} \end{bmatrix}$$

To summarize one must solve the following iteratively:

$$\begin{bmatrix} C_2 \\ C_3 \end{bmatrix}_{n+1} = \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}_n - T^{-1}(\vec{C}_n) \cdot \vec{\tilde{R}}(\vec{C}_n)$$

where $T^{-1}(C_n)$ is given above and

$$\vec{\tilde{R}}(\vec{C}_n) = \tilde{K}(\vec{C}_n)\,\vec{C}_n - \vec{\tilde{F}}$$

$$= \frac{AE_0}{h^2} \begin{bmatrix} 2h-C_3 & -(h+C_2-C_3) \\ -(h+C_2-C_3) & h+C_2-C_3 \end{bmatrix}_n \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}_n - \begin{bmatrix} 0 \\ R \end{bmatrix}$$

Let $A = 0.01\ m^2$, $L = 2m$, $E_0 = 100$ MPa $= 100 \times 10^6$ Pa

$R = 100$ KN $= 100 \times 10^3$ N. Since there are only two elements $h = 1$ m. Starting with a zero guess

$$\vec{C}_0 = \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \tilde{K}(\vec{C}_0) = 10^6 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = T(\vec{C}_0)$$

Then the Newton iterations yield:

$$\vec{C}_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \quad \tilde{K}(\vec{C}_1) = 10^6 \begin{bmatrix} 1.8 & -0.9 \\ -0.9 & 0.9 \end{bmatrix} \quad T(\vec{C}_1) = 10^6 \begin{bmatrix} 1.6 & -0.8 \\ -0.8 & 0.8 \end{bmatrix}$$

$$\vec{C}_2 = \begin{bmatrix} 0.1125 \\ 0.2250 \end{bmatrix} \quad \tilde{K}(\vec{C}_2) = 10^6 \begin{bmatrix} 1.775 & -0.8875 \\ -0.8875 & 0.8875 \end{bmatrix} \quad \tilde{T}(\vec{C}_2) = 10^6 \begin{bmatrix} 1.55 & -0.775 \\ -0.775 & 0.775 \end{bmatrix}$$

$$\vec{C}_3 = \begin{bmatrix} 0.1127 \\ 0.2254 \end{bmatrix} \quad \tilde{K}(\vec{C}_3) = 10^6 \begin{bmatrix} 1.7746 & -0.8873 \\ -0.8873 & 0.8873 \end{bmatrix} \quad \tilde{T}(\vec{C}_3) = 10^6 \begin{bmatrix} 1.5492 & -0.7746 \\ -0.7746 & 0.7746 \end{bmatrix}$$

$$\vec{C}_4 = \begin{bmatrix} 0.1127 \\ 0.2254 \end{bmatrix}$$

We stop here as the solution seems to have converged. We use this solution for post processing.