MATH 493 (Homework 4)

Finite Element Methods (Summer 2008)

1. Consider solving the following Boundary Value Problem (BVP) using the finite element method:

$$
\begin{aligned}
-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + b(x)u(x) &= f(x) \qquad\qquad 0 < x < 1 \\
u(0) &= 0 \\
u(1) &= 0
\end{aligned}
$$

   (a) Derive the continuous weak-variational form of the BVP.

   (b) Consider the finite element discretization of the given problem. Let the domain [0, 1] be partitioned into a finite number of sub-intervals M with nodes $x_1 = 0, x_2, \ldots, x_N = 1$. Let $u_h(x)$ be the finite element approximation to the exact solution $u(x)$ and let $u_h(x) = \sum_{i=1}^{N} c_i\phi_i(x)$ where $\phi_i(x)$ is the finite element hat function defined at node x_i. Starting from the discrete weak-variational form (corresponding to part(a)), derive the following matrix system:

$$(K + M)\ \vec{c} = \vec{f}$$

   where we have for $i, j = 1, \ldots, N$,

$$
\begin{aligned}
K_{ij} &= \int_0^1 a(x)\frac{d\phi_j}{dx}(x)\ \frac{d\phi_i}{dx}(x)\ dx \\
M_{ij} &= \int_0^1 b(x)\phi_j(x)\ \phi_i(x)\ dx \\
f_j &= \int_0^1 f(x)\phi_j(x)\ dx
\end{aligned}
$$

   Here K is the global *stiffness* matrix, M is the global *mass* matrix, $\vec{f}$ is the global *load* vector and $\vec{c}$ is the vector of unknown coefficients.

   (c) Let $a(x) = x,\ b(x) = 1,\ f(x) = x^2 - 5x + 1$. Consider partitioning the domain $[0, 1]$ into 3 elements ($M = 3$) and 4 nodes ($N = 4$). By defining the appropriate basis functions over these elements, setup the matrix system in part (b) and solve for the unknown coefficents. Determine $u_h(0.5)$ and compare the approximate value obtained with the exact solution $u(0.5)$.

   (d) Implement a one-dimensional finite element program for the given boundary value problem with $a(x) = x,\ b(x) = 1,\ f(x) = x^2 - 5x + 1$. Plot the exact solution $u(x)$ against the finite element solution $u_h(x)$ in the interval [0, 1] for the cases M=3, 6, 12 and explain your observation.

   (e) Write a subroutine to evaluate the error in the L_2-norm between the exact and the finite element solution over the domain. Perform a convergence study.