ANSWER KEY

Mathematics 108, Introductory Calculus **Spring 2009**
Test 1, Sections 1.4-1.6, questions from old tests

1. Consider the function: $f(x)=\dfrac{x^2+x-6}{2x^2-6x+4}$. Find each of the following limits, if the limit exists. If the limit does not exist, write DNE.

   a) $\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{(x+3)(x-2)}{2(x^2-3x+2)} = \lim_{x\to 2}\dfrac{(x+3)\cancel{(x-2)}}{2\cancel{(x-2)}(x-1)} = \dfrac{5}{2(1)} = \boxed{\dfrac{5}{2}}$

   Plug in: $\dfrac{4+2-6}{2(4)-12+4} = \dfrac{0}{0}$: keep going

   b) $\lim_{x\to 1} f(x) = \boxed{\text{DNE (vertical Asymptote)}}$

   plug in: $\dfrac{1+1-6}{2-6+4} = \dfrac{-4}{0}$

   c) $\lim_{x\to -3} f(x) = \dfrac{0}{4} = \boxed{0}$

   plug in: $\dfrac{9-3-6}{2(9)-18+4} = \dfrac{0}{4} = \boxed{0}$

   d) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\dfrac{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{6}{x^2}}{\frac{2x^2}{x^2}-\frac{6x}{x^2}+\frac{4}{x^2}} = \lim_{x\to\infty}\dfrac{1+\frac{1}{x}-\frac{6}{x^2}}{2-\frac{6}{x}+\frac{4}{x^2}} = \boxed{\dfrac{1}{2}}$ ($\frac{1}{x}\to 0$, $\frac{6}{x^2}\to 0$, $\frac{6}{x}\to 0$, $\frac{4}{x^2}\to 0$)

2. Find the following limits, if the limits exist. If the limit does not exist, write DNE.

   a) $\lim_{x\to\infty}(x^2+2x+7) = +\infty$ (limit DNE: quadratic function, grows without bounds)

   b) $\lim_{x\to\infty}\dfrac{3-x^2}{4x^2+5} = \lim_{x\to\infty}\dfrac{\frac{3}{x^2}-\frac{x^2}{x^2}}{\frac{4x^2}{x^2}+\frac{5}{x^2}} = \lim_{x\to\infty}\dfrac{\frac{3}{x^2}-1}{4+\frac{5}{x^2}} = \boxed{-\dfrac{1}{4}}$ ($\frac{3}{x^2}\to 0$, $\frac{5}{x^2}\to 0$)

   c) $\lim_{x\to\infty}\dfrac{3x^2-7}{x^4-1} = \lim_{x\to\infty}\dfrac{\frac{3x^2}{x^4}-\frac{7}{x^4}}{\frac{x^4}{x^4}-\frac{1}{x^4}} = \dfrac{\frac{3}{x^2}-\frac{7}{x^4}}{1-\frac{1}{x^4}} = \dfrac{0}{1} = \boxed{0}$ ($\frac{3}{x^2}\to 0$, $\frac{7}{x^4}\to 0$, $\frac{1}{x^4}\to 0$)

3. Given the graph of $g(x)$, find (if they exist)

   a) $\lim_{x\to 0^+} g(x) =$ 3

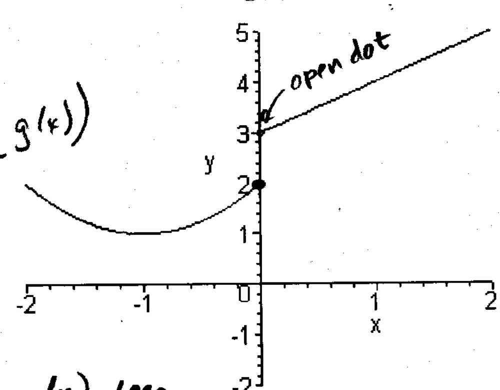
   b) $\lim_{x\to 0^-} g(x) =$ 2

   c) $\lim_{x\to 0} g(x) =$ DNE (because $\lim_{x\to 0^+} g(x) \neq \lim_{x\to 0^-} g(x)$)

   d) $g(0) =$ 2

   e) Is $g(x)$ continuous at $x=0$? NO
   Briefly explain why or why not, using mathematical concepts in your answer.

$g(x)$



$g(x)$ is not continuous at $x=0$ because $\lim_{x\to 0} g(x)$ does not exist. NOTE: it is continuous everywhere else!

4. Is the function $f(x)$ below continuous at $x = 3$? $\boxed{\text{Yes}}$ Justify your answer mathematically (not just with a graph).

① find $\lim_{x \to 3^-} f(x)$ & $\lim_{x \to 3^+} f(x)$.

$$f(x) = \begin{cases} x^2 + 3, & if\ x \leq 3 \\ 4x, & if\ x > 3 \end{cases}$$

- $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 + 3 = 3^2 + 3 = 12$
- $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 4 \cdot x = 12$

Solution:
I. $\lim_{x \to 3} f(x) = 12 = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$
II. $f(3) = 3^2 + 3 = 12$
III. $\lim_{x \to 3} f(x) = f(3)$
$\therefore$ continuous at $x = 3$

5. List **all** values, if any, at which the following functions are discontinuous. Describe any discontinuities you find.

a) $f(x) = x^5 - 15x^4 + 9x - 11$ No discontinuities (polynomials are continuous everywhere)

b) $g(x) = \dfrac{x}{2x^2 - 8x}$

let $2x^2 - 8x = 0$
$2x(x-4) = 0$
$x = 0 \quad x = 4$

$\boxed{\therefore \text{Discontinuities at } x = 0,\ x = 4.}$

Q: To see what kind, check the limits:

$\lim_{x \to 0} \frac{x}{2x^2 - 8x} = \lim_{x \to 0} \frac{\cancel{x}^{\,1}}{\cancel{x}(2x-8)} = \frac{1}{2 \cdot 0 - 8} = \boxed{-\frac{1}{8}}$ "Hole" or Removable discontinuity

$\lim_{x \to 4} \frac{x}{2x^2 - 8x}$ = DNE (plug in: get $\frac{4}{0}$) ← vertical asymptote.

6. A bookstore has been offering a special commemorative book at a price of \$15 per book, and at that price, has been selling 24 books per month. The bookstore is planning to reduce the price to stimulate sales and estimates that for each \$1 reduction in price, 8 more books will be sold each month.

a) Find the linear demand function that models the facts above. Express the demand ($D(p)$) for the book as a function of the price p at which the book is sold.

$\frac{\Delta D}{\Delta p} = \frac{+8 \text{ (more books)}}{-1 \text{ (less cost)}}$

$D - 24 = -8(p - 15) = -8p + 120$

$\boxed{D = -8p + 144}$

b) Express the total revenue which the bookstore will receive as a result of the sales of the commemorative book as a function of the price p of the books.

$R(p) = p \cdot D(p) = p(-8p + 144) = -8p^2 + 144p$

The bookstore can obtain the book from the publisher at a cost of \$6 per book.

a) Express the total profit which the bookstore can make on the sale of the books as a function of the price p of the books.

Profit $= R(p) - C(p) = \underbrace{-8p^2 + 144p}_{R(p)} - \underbrace{6}_{\text{cost per item}}\underbrace{(-8p + 144)}_{D(p)\ (\#\text{ of items})}$

$= -8p^2 + 144p + 48p - 6 \cdot 144$
$= -8p^2 + 192p - 864$
$= -8(p^2 - 24p + 108)$

b) At what price should the books be sold to generate the greatest profit? \$12.00
What will the store's total profit be at that price? \$288

$p = \frac{-b}{2a} = \frac{-192}{2(-8)} = \frac{-192}{-16} = \$12$

$P(p) = -8(144 - 24(12) + 108)$
$= -8(-36) = \boxed{\$288}$

c) How do you know that profit is maximized at that point?

Because $P(p) = -8p^2 + 192p - 864$ is a parabola opening down ($a = -8 < 0$), so the vertex is a maximum. Coordinates of the vertex are $(12, 288)$