ANSWER KEY, Review for Chapter 3 (Spring 2009)

1. $f(x) = \frac{x}{x^2+x+1}$ (function 1)

a) Identify Critical Points.

$f'(x) = \frac{1(x^2+x+1) - x(2x+1)}{(x^2+x+1)^2}$ $f = x$, $g = x^2+x+1$, $f' = 1$ $g' = 2x+1$

$= \frac{x^2+\cancel{x}+1-2x^2-\cancel{x}}{(x^2+x+1)^2}$

$= \frac{-x^2+1}{(x^2+x+1)^2}$. To find critical values, let $-x^2+1=0$

$(1-x)(1+x)=0$

$x=1$ $x=-1$ ← critical values

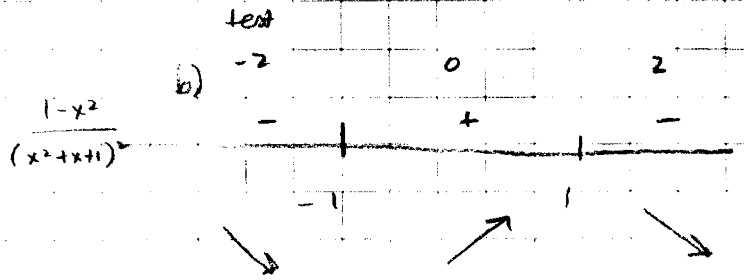
(Note: there are no points of discontinuity. How do we know? Let $x^2+x+1=0$. Use quadratic formula: $x = \frac{-1 \pm \sqrt{1^2-4(1)(1)}}{2(1)}$

$= \frac{-1 \pm \sqrt{-3}}{2}$. The negative number under the radical tells you that there are no real solutions to $x^2+x+1=0$. Thus, there are no values of x to omit, and the domain of f(x) is all reals.)

Critical points: $f(1) = \frac{1}{1+1+1} = \frac{1}{3} \Rightarrow (1, 1/3)$ is 1 point

$f(-1) = \frac{-1}{(-1)^2-1+1} = \frac{-1}{1} = -1 \Rightarrow (-1,-1)$ is 2nd point.

b)



(-1, -1) is a relative minimum.
(1, 1/3) is a relative maximum.

c) Increase: (-1, 1) OR $-1 < x < 1$
Decrease $(-\infty, -1) \cup (1, +\infty)$ OR $x < -1$ or $x > 1$

(function 2)

1. $f(x) = \frac{x^2+5x+6}{x^2-4}$

a) We can simplify $f(x)$ before we find derivative (read NOTE →

$f(x) = \frac{x^2+5x+6}{x^2-4} = \frac{(\cancel{x+2})(x+3)}{(\cancel{x+2})(x-2)}$

$f'(x) = \frac{1(x-2) - 1(x+3)}{(x-2)^2}$

$= \frac{\cancel{x}-2-\cancel{x}-3}{(x-2)^2} = \frac{-5}{(x-2)^2}$

$f = x+3 \quad g = x-2$

$f' = 1 \quad g' = 1$

Domain: let $x^2-4=0$

$(x+2)(x-2)=0$

$x=-2 \quad x=2$ $\therefore$ Domain of $f(x)$ is $\{x \mid x \in \mathbb{R}, x \neq -2, x \neq 2\}$

NOTE: This problem has a twist.

If we find $\lim_{x \to 2} f(x)$, $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{4+10+6}{0} = \frac{20}{0}$.

The limit DNE, and there is a vertical asymptote at $x = 2$.

BUT, $\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{(\cancel{x+2})(x+3)}{(\cancel{x+2})(x-2)} = \boxed{\frac{1}{-4}}$.

If the limit exists, there is NOT a V.A. at that point. We just exclude $x=-2$ from the domain and work with the simplified function.

This function has NO critical points. (because $-5 \neq 0$. it's constant)

b. N/A

c. The derivative of the function is negative everywhere, so the function is decreasing everywhere on its domain. (-5 is always negative; $(x-2)^2$ is always positive; their quotient is negative.)

1. $f(x) = (x^2-4)^2$ (function 3) Domain: $\mathbb{R}$ (polynomial function)

a) $f'(x) = 2(x^2-4)^1(2x) = 4x(x^2-4)$ (chain rule)

CV: let $4x(x^2-4) = 0 = 4x(x+2)(x-2)$

$4x = 0 \quad x+2=0 \quad x-2=0$

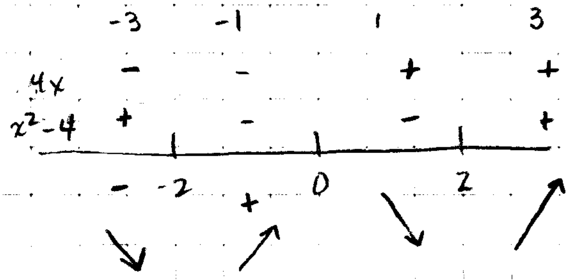
$x = 0 \quad x = -2 \quad x = 2$

Crit pts: $f(0) = 16$: $(0,16)$

$f(-2) = 0$: $(-2,0)$

$f(2) = 0$: $(2,0)$

b)



$(-2,0)$ is a rel. minimum

$(0,16)$ is a rel. maximum

$(2,0)$ is a rel. minimum

c) Intervals of increase: $(-2,0) \cup (2,+\infty)$

Intervals of decrease: $(-\infty,-2) \cup (0,2)$

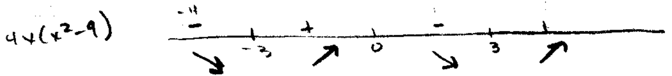
2. Consider the function $g(x) = (x^2 - 9)^2$.

a) Identify all Critical values of $g(x)$.

$g'(x) = 2(x^2-9)(2x)$ set $g'(x) = 0 = 4x(x-3)(x+3)$

CV's: $\boxed{x=0 \quad x=3 \quad x=-3}$

b) Identify the intervals of increase and decrease of $g(x)$.



Decrease: $(-\infty,-3) \cup (0,3)$
Increase: $(-3,0) \cup (3,+\infty)$

Note: you must specify the intervals (not just draw picture) for full credit.

c) Classify your critical values as relative maxima or relative minima.

x = −3: rel. minimum   x = 3: rel. minimum

x = 0: rel. maximum

d) Find the absolute maximum and absolute minimum of $g(x)$ on the interval $[-1, 5]$. Be sure to show your reasoning.

Find g's at CV's and endpoints:
$g(-1) = (1-9)^2 = 8^2 = 64$
$g(5) = (25-9)^2 = 16^2 = 256$ − max
$g(-3) = (9-9)^2 = 0$ − min
$g(0) = (-9)^2 = 81$
$g(3) = (9-9)^2 = 0$ − min

Absolute max: (5, 256)
absolute min: (−3, 0) and (3, 0)

3. Find the absolute maximum and absolute minimum (if any) of the function $f(x) = \frac{1}{4}x^4 - 18x^2$ on the interval $-10 \le x \le 5$. (Be certain to give both coordinates of the points you find.) → Note: This equals $\frac{1}{4} \cdot x^4 - 18x^2$

1. $f'(x) = \frac{1}{4} \cdot 4 \cdot x^3 - 36x$
$= x^3 - 36x \quad = x(x^2-36) \quad = x(x+6)(x-6)$
Let $f'(x) = 0$. Then $x = 0$, $x = -6$, $x = 6$

Because we are looking for abs. max/min, we don't have to do 1st derivative test.
Just find: $f(-10) = \frac{1}{4}(-10)^4 - 18(100) = 2500 - 1800 = 700$
endpts. → $f(5) = \frac{1}{4}(625) - 18(25) = 156.25 - 450 = -293.75$
CV's $\begin{cases} f(0) = 0 \\ f(6) = \frac{1}{4}(6^4) - 18(36) = -324 \\ f(-6) = \frac{1}{4}(-6)^4 - 18(-6)^2 = -324 \end{cases}$

Absolute maximum at (−10, 700)

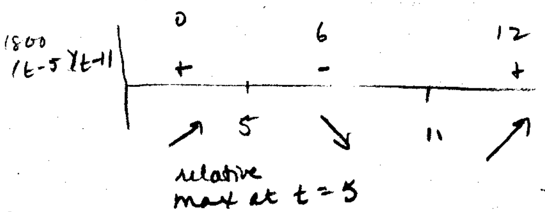
Absolute minimum at (6, −324) and (−6, 324)

(P.S. I won't give numbers this nasty)

4. Suppose that t years after its founding in 1990, a certain national organization had a membership of $M(t) = 300(2t^3 - 48t^2 + 330t)$.

a) In what year between 1990 and 2003 was the membership of the organization the largest? $0 \le t \le 13$

$M'(t) = 300(6t^2 - 96t + 330)$
$= 1800(t^2 - 16t + 55)$
$= 1800(t-5)(t-11)$
$\underbrace{t = 5 \quad t = 11}_{\text{2 critical values}}$



Note: not necessary, because of plugging in values below.

Check endpoints and CV's: $M(0) = 0$
$M(5) = 300(2 \cdot 5^3 - 48 \cdot 5^2 + 330 \cdot 5)$
$300(250 - 1200 + 1650)$
$= 300 \cdot 700 = 210{,}000$
$M(11) = 300(2 \cdot 11^3 - 48 \cdot 11^2 + 330 \cdot 11) = 484 \cdot 300 = 145{,}200$
$M(13) = 300(2 \cdot 13^3 - 48 \cdot 13^2 + 330 \cdot 13) = 300 \cdot 572 = 171{,}600$

∴ Membership was largest in 1995 (t = 5)

b) What was the membership at that time?

Membership was 210,000

c) In what year between 1990 and 2003 was the membership the smallest?

If we include the endpoints, membership was smallest in the first year ($t=0$).

d) What was the membership in that year?

There were no members that year.

5. Let $p=(28-7x)$ for $0 \le x \le 4$ be the price per unit at which x thousand units of a certain commodity will be sold, and let $R(x)=x \cdot p$ be the revenue obtained from the sale of the x thousand units.

a) Find the Marginal Revenue $R'(x)$. $R(x) = x(28-7x) = 28x - 7x^2$

$R'(x) = 28 - 14x$

b) For what level of production is the revenue maximized?

Let $R'(x) = 0 = 28 - 14x$; $14x = 28$; $x = 2$.

- Revenue is maximized at $x=2$, so 2 thousand units.

Is it a maximum? Only one CV & $R''(x) = -14 < 0$, so $x=2$ is an abs. max.

c) If the cost to produce each unit is \$3.50 per unit, find the profit function $P(x)=R(x)-C(x)$, where $C(x)$ is the total cost to produce x thousand units.

$P(x) = 28x - 7x^2 - 3.50x = 24.50x - 7x^2$

(remember that x is quantity)

d) For what level of production is profit maximized?

Profit will be maximized when $P'(x) = 0 = 24.5 - 14x$; $14x = 24.5$; $x = \frac{24.5}{14} = 1.75$ thousand units.

x is in thousand units.

Max? $P''(x) = -14$, so $x = 1.75$ is an absolute max.

6. A company determines that if x thousand dollars are spent on advertising a certain product, then $S(x)$ units of the product will be sold, where:

$$S(x) = -2x^3 + 27x^2 + 132x + 207 \qquad for\ 0 \le x \le 17.$$

a) How many units will be sold if nothing is spent for advertising?

If nothing is spent, $x=0$, and $S(x) = 0 + 207 = 207$ units

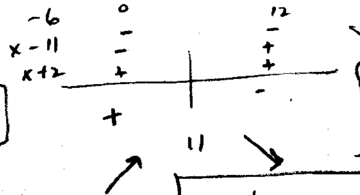
b) How much should be spent on advertising to maximize sales?

$S'(x) = -6x^2 + 54x + 132 = -6(x^2 - 9x - 22) = -6(x-11)(x+2)$

$x = 11$ $x = -2$ ← we wont spend -2 thousand dollars.

c) What is the maximum sales level?

Q: Is $x = 11$ a relative max?



yes, it is a relative max, only max, for $x \ge 0$.

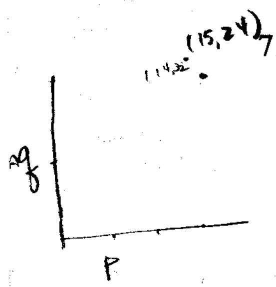
So \$11,000 should be spent

units are thousand dollars

Maximum sales level will be

$S(11) = -2(11)^3 + 27(11)^2 + 132(11) + 207 = \boxed{2264 \text{ units}}$

$\frac{\Delta q}{\Delta p} = \frac{8}{-1}$ increase in books / reduction in price



7. A bookstore has been offering a special commemorative book at a price of \$15 per book, and at that price, has been selling 24 books per month. The bookstore is planning to reduce the price to stimulate sales and estimates that for each \$1 reduction in price, 8 more books will be sold each month.

a) Find the linear demand function that models the facts above. Express the demand = also q (*D*(*p*)) for the book as a function of the price *p* at which the book is sold.

$q-24=-8(p-15)$ (point-slope), $q-24=-8p+120$; $\boxed{q=-8p+144}$

b) Express the total revenue which the bookstore will receive as a result of the sales of the commemorative book as a function of the price *p* of the books.

$R(p)=p(-8p+144)=-8p^2+144p$

The bookstore can obtain the book from the publisher at a cost of \$6 per book. $q=-8p+144$

c) Express the total profit which the bookstore can make on the sale of the books as a function of the price *p* of the books.

$P(p)=-8p^2+144p-6(-8p+144)=-8p^2+144p+48p-864=-8p^2+192p-864$

d) Use calculus to determine the price at which the books should be sold to generate the greatest profit? \$12. What will the store's total profit be at that price?

$P'(p)=-16p+192$ Let $-16p+192=0$; $-16p=192$; $p=12$. Total profit $=-8(12)^2+192(12)-864=$ $\boxed{\$288}$

e) How do you know that profit is maximized at that point?

$P''(p)=-16$. $P(p)$ is continuous everywhere; there is only one C.V., and $P''(p)<0$, so the critical value must be a maximum.

8. A postal clerk comes to work at 7 a.m., and *t* hours later has sorted approximately

$$g(t)=\frac{-t^3}{2}+3t^2+100t \text{ letters.} = -\tfrac{1}{2}\cdot t^3+3t^2+100t$$

a) At what time during the period from 7 a.m. to noon is the clerk performing at peak efficiency? Need to find g'' (t) and set it = 0. → Domain: $0\le t\le 5$

$g'(t)=-\tfrac{3}{2}t^2+6t+100$ Let $-3t+6=0$; $3t=6$; $t=2$; so at 9:00 am he is most efficient.

$g''(t)=-3t+6$

b) At what **rate** (letters per hour) is he sorting letters at that time?

$g'(t)$ (derivative is a rate). $g'(2)=-\frac{3}{2}(2^2)+6\cdot2+100=$
$-6+12+100=106$ letters / hr.

9. $3x^2+2y^2=12$ (use implicit or explicit differentiation)

$\frac{dy}{dx}=y'=\boxed{\frac{-3x}{2y}}$ $\left(\text{or } -\frac{3x}{2}\cdot\left(\frac{12-3x^2}{2}\right)^{-1/2}\right)$

Implicit: $6x+4y\frac{dy}{dx}=0$;

$4y\frac{dy}{dx}=-6x$; $\frac{dy}{dx}=\frac{-6x}{4y}=\frac{-3x}{2y}$

OR Explicit $2y^2=12-3x^2$

$y^2=\frac{12-3x^2}{2}$

$y=\sqrt{\frac{12-3x^2}{2}}$

$y'=\frac{1}{2}\left(\frac{12-3x^2}{2}\right)^{-1/2}\cdot\left(\frac{1}{2}(-6x)\right)$

$=\frac{-3x}{2}\left(\frac{12-3x^2}{2}\right)^{-1/2}$

10. $x^3+2xy-y^4=7$ (use implicit differentiation)

$\frac{dy}{dx}=y'=\frac{3x^2+2y}{4y^3-2x}$

$f=2x \quad g=y$
$f'=2 \quad g'=y'$

$3x^2+2x\cdot y'+2y-4y^3y'=0$

$3x^2+2y=4y^3y'-2xy'=y'(4y^3-2x)$

$y'=\frac{3x^2+2y}{4y^3-2x}$