ANSWER KEY

Mathematics 108 , Spring 2009
Chapter 2, questions from old tests

Note: limits are required.

1. State the formal definition of the derivative function, $f'(x)$. $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

2. Let $f(x)=3x^2-4x+2$.
   a) What is the average rate of change of $f(x)$ between $x=0$ and $x=1$?

$$\text{Avg rate}=\frac{f(1)-f(0)}{1-0}=\frac{3(1)^2-4(1)+2-(3\cdot 0^2-4\cdot 0+2)}{1}=\frac{3-4+2-2}{1}=\frac{-1}{1}=-1$$

   b) Using the formal definition of the derivative function, find the derivative of the function $f(x)=3x^2-4x+2$. NOTE: you must show all steps of your work, *neatly*, in order to receive any credit. Showing an answer only or using derivative shortcuts will result in no credit for this problem.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{3(x+h)^2-4(x+h)+2-(3x^2-4x+2)}{h}$$

$$=\lim_{h\to 0}\frac{3(x^2+2xh+h^2)\cancel{-4x}-4h\cancel{+2}-3x^2\cancel{+4x}\cancel{-2}}{h}=\lim_{h\to 0}\frac{\cancel{3x^2}+6xh+3h^2-4h\cancel{-3x^2}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(6x+3h-4)}{\cancel{h}}=\lim_{h\to 0}6x+\underbrace{3h}_{\to 0}-4=\boxed{6x-4}$$

   c) What is the instantaneous rate of change of $f(x)$ at $x=1$? $f'(1)=6(1)-4=2$.

For problems 3-6 , find the first derivative of each function, using any <u>correct</u> technique. Unless otherwise indicated, you do not need to simplify your answer.

3. $f(x)=6x^3+\frac{4}{x^7}+5\sqrt{x^3}+\sqrt{5}=$ $\qquad f'(x)=$ $18x^2-28x^{-8}+\frac{15}{2}x^{1/2}$

   $f(x)=6x^3+4x^{-7}+5x^{3/2}+\sqrt{5}$

4. $f(x)=\frac{2x^3+3x}{x^2-4x}=\frac{\cancel{x}(2x^2+3)}{\cancel{x}(x-4)}$ $\qquad f'(x)=$ $\frac{2x^2-16x-3}{(x-4)^2}$ or $\frac{2x^4-16x^3-3x^2}{(x^2-4x)^2}$ (without simplifying)

   (combine like terms in the numerator)

Quotient Rule (simplified function)

$f=2x^2+3 \quad g=x-4$

$f'=4x \quad g'=1$

$$f'(x)=\frac{4x(x-4)-1(2x^2+3)}{(x-4)^2}=\frac{4x^2-16x-2x^2-3}{(x-4)^2}=\frac{2x^2-16x-3}{(x-4)^2}$$

Non-simplified function:

$f=2x^3+3x \quad g=x^2-4x$

$f'=6x^2+3 \quad g'=2x-4$

$$f'(x)=\frac{(6x^2+3)(x^2-4x)-(2x^3+3x)(2x-4)}{(x^2-4x)^2}=\frac{6x^4-24x^3+3x^2\cancel{-12x}-4x^4+8x^3-6x^2\cancel{+12x}}{(x^2-4x)^2}$$

5. $f(x)=\dfrac{2}{(4x^3+7)^5}$ $=2(4x^3+7)^{-5}$ (Rewrite) $f'(x)=$ $\dfrac{-120x^2(4x^3+7)^4}{}$ $=$

$\dfrac{0-120x^2(4x^3+7)^4}{(4x^3+7)^{10}}$ } Quotient Rule

$f'(x)=-10\cdot(4x^3+7)^{-6}(12x^2)$

chain Rule

OR Quotient

$f=2$ $g=(4x^3+7)^5$

$f'=0$ $g'=5(4x^3+7)^4(12x^2)=60x^2(4x^3+7)^4$

6. $f(x)=(\sqrt{x}+4)(x^3-7x)$ (product Rule) $f'(x)=$ $\boxed{\frac{1}{2}x^{-1/2}(x^3-7x)+(x^{1/2}+4)(3x^2-7)}$

$f=x^{1/2}+4$ $g=x^3-7x$

$f'=\frac{1}{2}x^{-1/2}$ $g'=3x^2-7$

In problems 7 and 8, find the first and second derivatives of the indicated function:

7. $f(x)=x^4-3x^2+2x$ $f'(x)=$ $4x^3-6x+2$

$f''(x)=$ $12x^2-6$

8. $f(x)=(6x^3-4x)^5$ $f'(x)=$ $5(6x^3-4x)^4(18x^2-4)$ (chain Rule)

use product rule!

$f=5(6x^3-4x)^4$ $g=18x^2-4$ $f''(x)=$ $20(6x^3-4x)^3(18x^2-4)^2+180x(6x^3-4x)^4$

$f'=20(6x^3-4x)^3(18x^2-4)$, $g'=36x$

9. Find an equation of the line tangent to $g(x)=(x^2)(x^4-1)^3$ at $x=1$.

① Point: $g(1)=1^2(1^4-1)^3=1\cdot0=0$ ∴ $(1,0)$ is the point.

② Derivative: $g'(x)$ (needs product Rule)

$f=x^2$ $g=(x^4-1)^3$

$f'=2x$ $g'=3(x^4-1)^2(4x^3)$

$=12x^3(x^4-1)^2$

$g'(x)=2x(x^4-1)^3+x^2(12x^3)(x^4-1)^2$

$g'(1)=1\cdot0^3+1\cdot12\cdot0^2=0+0=0$

Line: $y-0=0(x-1)$ $\Rightarrow$ $\boxed{y=0}$ (horizontal line - actually, the x-axis)

10. When coffee makers are sold for p dollars apiece, the monthly demand for coffee makers, (that is, the number of coffee makers local consumers will buy) is given by the function $D(p)=\dfrac{60{,}000}{2p+4}$ coffee makers. It is estimated that t months from now, the unit price of a coffee maker will be $p(t)=56-2\sqrt{t}$ dollars. Compute the rate of change of monthly demand for coffee makers with respect to time 16 months from now. (Looking for $\dfrac{dD}{dt}$) demand time

2 solutions: ① Compose functions at the beginning:

$D(t)=\dfrac{60000}{2(56-2\sqrt{t})+4}=\dfrac{60000}{112-4\sqrt{t}+4}=\dfrac{60000}{116-4\sqrt{t}}=60{,}000(116-4\sqrt{t})^{-1}$ (composed function)

$D'(t)=-60000(116-4\cdot t^{1/2})^{-2}(-4\cdot\frac{1}{2}\cdot t^{-1/2})=-60000(116-4\cdot t^{1/2})^{-2}(-2t^{-1/2})$

chain rule

$D'(16)=\dfrac{-60{,}000}{(116-4\sqrt{16})^2}\cdot\dfrac{-2}{\sqrt{16}}=\dfrac{120{,}000}{(100)^2\cdot4}=\dfrac{12\cancel{0{,}000}}{1\cancel{0000}\cdot4}=3$ coffeemakers/mo.

$116-16$

or ② Use Chain Rule: $\frac{dD}{dt} = \frac{dD}{dp} \cdot \frac{dp}{dt}$

a) Rewrite: let $D(p) = 60{,}000(2p+4)^{-1}$

$$\frac{dD}{dp} = 60{,}000(-1)(2p+4)^{-2}(2) = \frac{-120{,}000}{(2p+4)^2}$$

b) $\frac{dp}{dt} = 0 - 2 \cdot \frac{1}{2} \cdot t^{-1/2} = \frac{-1}{\sqrt{t}}$

c) at $t = 16$, $p = 56 - 2\sqrt{16} = 56 - 2 \cdot 4 = 56 - 8 = 48$

So at $t = 16$, $p = 48$ and $\frac{dD}{dt} = \underbrace{\frac{-120{,}000}{(2\cdot 48+4)^2}}_{p} \cdot \underbrace{\frac{-1}{\sqrt{16}}}_{t} = \frac{120{,}000}{(96+4)^2 \cdot 4} = \frac{\overset{30{,}000}{\cancel{120{,}000}}}{100^2 \cdot \cancel{4}} =$

(keep variables consistent)

$\frac{30{,}000}{10{,}000} = 3$ coffeemakers/mo

11. An entrepreneurial tee-shirt maker estimates that his **total cost** of producing q shirts is $C(q) = \frac{1}{4}q^2 + 2q + 40$ and that he can sell all q shirts if the price **per shirt** is $p(q) = 28 - q$.

a) Find the marginal cost function.

Marginal cost: $C'(q) = \frac{1}{2} \cdot q + 2 = \frac{q}{2} + 2$

b) Find the revenue function, $R(q)$, and find the marginal revenue function.

Revenue: $R(q) =$ price × quantity $= (28-q)q = 28q - q^2$. Marginal Revenue: $28 - 2q$

c) Find the profit function as a function of quantity, and find the marginal profit function.

Profit = Revenue − cost $= 28q - q^2 - (\frac{1}{4}q^2 + 2q + 40) = 28q - q^2 - \frac{1}{4}q^2 - 2q - 40 = \underbrace{-\frac{5}{4}q^2 + 26q - 40}_{\text{Profit}} = P(q)$

d) Use the functions above to estimate:

Marginal profit $= P'(q) = -\frac{5}{2}q + 26$

i) The cost of producing the fifth tee-shirt. ($= C'(4)$)

$C'(4) = \frac{4}{2} + 2 = 2 + 2 = \$4.00$/shirt

ii) The profit from the sale of the fifth tee-shirt ($= P'(4)$)

$P'(4) = -\frac{5}{2}(4) + 26 = -10 + 26 = \$16$/shirt