Mathematics 108
Chapter 2 Test

Name: ANSWER KEY B
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No calculators allowed. Show all work neatly. Please put your answers in the spaces provided or in $\boxed{\text{boxes}}$

1. Use the limit definition of the derivative, that is $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, to find the derivative of $f(x) = 3x - 2x^2$. You may check your answer with the power rule, but you must show all steps of your reasoning to get any credit for this answer.

$f'(x) = \lim_{h\to 0} \frac{3(x+h) - 2\overset{x^2+2xh+h^2}{(x+h)^2} - (3x-2x^2)}{h} = \lim_{h\to 0} \frac{\cancel{3x}+3h - \cancel{2x^2} - 4xh - 2h^2 - \cancel{3x} + \cancel{2x^2}}{h} =$

$\lim_{h\to 0} \frac{3h-4xh-2h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(3-4x-2h)}{\cancel{h}} = \lim_{h\to \infty} 3-4x-\cancel{2h}^{\to 0} = \boxed{3-4x}$

2. Find the following derivatives, using any correct techniques. Unless otherwise specified, you do not have to simplify your answers.

a) $f(x) = \left(4 - \frac{1}{x^2}\right)\left(x^2 - 3x\right)$ $\qquad f'(x) = \underset{\text{expand first}}{8x - 12 - 3x^{-2}}$ OR

I. Expand: $f(x) = 4x^2 - 12x - 1 + 3x^{-1}$

$f'(x) = 8x - 12 - 3x^{-2}$

OR $f = 4 - x^{-2}$ $\quad g = x^2 - 3x$

$f' = 2x^{-3}$ $\quad g' = 2x - 3$

$f'(x) = 2x^{-3}(x^2-3x) + (4 - \frac{1}{x^2})(2x-3)$

b) $f(x) = \frac{6x-5}{x^2+1}$ $\qquad f'(x) = \frac{-6x^2+10x+6}{(x^2+1)^2} = \frac{-2(-3x^2+5x+3)}{(x^2+1)^2}$

(simplify the numerator)

$f = 6x-5$ $\quad g = x^2+1$

$f' = 6$ $\quad g' = 2x$

$f'(x) = \frac{6(x^2+1) - 2x(6x-5)}{(x^2+1)^2}$

$= \frac{6x^2+6-12x^2+10x}{(x^2+1)^2}$

c) $f(x) = \sqrt[3]{4x^2 - 5}$ $= (4x^2-5)^{1/3}$ $\qquad f'(x) = \frac{1}{3}(4x^2-5)^{-2/3}(8x)$ $= \frac{8x}{3}(4x^2-5)^{-2/3}$

$f'(x) = \frac{1}{3}(4x^2-5)^{-2/3}(8x)$

3. Find the first and second derivatives of each of the following:

a) $f(x) = 3x^2 + \frac{5}{x}$ $= 3x^2 + 5x^{-1}$

$f'(x) = 6x - 5x^{-2}$

$f''(x) = 6 + 10x^{-3}$

b) $f(x) = (x^3 - 1)^5$

$f'(x) = 5(x^3-1)^4(3x^2) = 15x^2(x^3-1)^4$

$f''(x) = 30x(x^3-1)^4 + 15x^2(12x^2)(x^3-1)^3$

$180x^4(x^3-1)^3 + 30x(x^3-1)^4$

$f = 15x^2$ $\quad g = (x^3-1)^4$

$f' = 30x$ $\quad g' = 4(x^3-1)^3(3x^2)$

$= 12x^2(x^3-1)^3$

$= 3x^2 - 4x^5$

4. Find an equation of the line tangent to $f(x) = x^2(3 - 4x^3)$ at $x = 1$.

3 1. Point: $f(1) = 1^2(3-4) = 1(-1) = -1$: $(1, -1)$

4 2. Derivative: $f'(x) = 2x(3-4x^3) - 12x^2(x^2)$ or $f'(x) = 6x - 20x^4$

4 3. Slope: $f'(1) = 2(3-4) - 12(1) = 2(-1) - 12 = -14$

3 4. Equation: $y - (-1) = -14(x-1)$ or $y + 1 = -14x + 14$ or $y = -14x + 13$

5. A fast food restaurant has determined that the relationship between the price, in dollars, at which it can sell hamburgers, $p(q)$, and the quantity q that it can sell is $p(q) = \dfrac{60{,}000 - q}{20{,}000}$

a) Find the revenue function as a function of the quantity of hamburgers sold (ie, find $R(q)$).

$$R(q) = q \cdot p = q\left(\frac{60{,}000 - q}{20{,}000}\right) = \frac{60000q - q^2}{20000} = \frac{1}{20000} \cdot (60{,}000q - q^2)$$

b) Find the marginal revenue function.

$$R'(q) = \frac{1}{20000} \cdot (60000 - 2q) = \frac{60{,}000 - 2q}{20{,}000}$$

c) Use the marginal revenue function to estimate the revenue from the sale of the 10,001st hamburger and interpret your result (tell me in words, with units, what your answer means.) $R'(10{,}000) = \dfrac{60{,}000 - 2(10000)}{20{,}000} = \dfrac{60{,}000 - 20000}{20{,}000} = \dfrac{40{,}000}{20000} = \$2$ / burger.

The sale of the 10,001st hamburger will produce revenue of about \$2.00.

6. When a company produces and sells x thousand units per week, its total weekly profit is P thousand dollars, where $P = \dfrac{200x}{100 + x^2}$.

10 pts.

The production level, in thousands of units, at t weeks from the present is $x = 4 + 2t$.

Find the function that models how fast profits are changing with respect to time (that is, find $\dfrac{dP}{dt}$).

Two approaches: I. $\dfrac{dP}{dt} = \dfrac{dP}{dx} \cdot \dfrac{dx}{dt} = \underbrace{\dfrac{200(100+x^2) - 2x(200x)}{(100+x^2)^2}}_{dP/dx} \cdot \underbrace{2}_{dx/dt} = $ (substitute $4+2t$ for x)

$\dfrac{20000 + 200x^2 - 400x^2}{(100+x^2)^2} = \dfrac{20000 - 400x^2}{}$

to find $\frac{dP}{dx}$: $\boxed{\begin{matrix} f = 200x & g = 100 + x^2 \\ f' = 200 & g' = 2x \end{matrix}}$

$$= \frac{200(100 + (4+2t)^2) - 2(4+2t)(200)(4+2t)}{(100 + (4+2t)^2)^2} \cdot 2$$

(Extra credit: What happens to profits in the long run? How do you know?)

or

II. Substitute first: $P = \dfrac{200(4+2t)}{100 + (4+2t)^2} = \dfrac{800 + 400t}{100 + 16 + 16t + 4t^2} = \dfrac{800 + 400t}{116 + 16t + 4t^2}$

$$\frac{dP}{dt} = \frac{400(116 + 16t + 4t^2) - (16 + 8t)(800 + 400t)}{(116 + 16t + 4t^2)^2}$$