Math 105 Fall 2009 ANSWER KEY

Final exam sample questions, from old tests, etc.

A note on this exam: it will be comprehensive and will require you to solve problems using information from the entire course. The questions may require you to use what you learned in several different contexts to solve a problem. Among the skills I expect you to be able to draw on in any context are:

1. Write the equation of a line.
2. Use laws of exponents correctly in any context.
3. Factor by factoring out greatest common factors, even if the common factor is a polynomial, exponential, log, or trig function.
4. Solve any quadratic, determining whether it has real roots and if so, whether the roots are rational or irrational.
5. Use the technique of completing the square.
6. Find function inverses, including function inverses of log and exponential functions

So here are some typical questions:

1. Find all real solutions to the following, using any mathematically correct technique.
   a) $y^2(y-7)-9(y-7)=0$ $=(y-7)(y^2-9)=(y-7)(y+3)(y-3)$. So: $y-7=0$; $y+3=0$; $y-3=0$. Solutions: $\boxed{y=7,\ y=-3,\ y=3}$
   b) $x^2+3x=-1$ $\Rightarrow x^2+3x+1=0$ Use Quadratic Formula: $x=\frac{-3\pm\sqrt{9-4(1)(1)}}{2}=\boxed{\frac{-3\pm\sqrt{5}}{2}}$
2. Find all solutions to the following inequalities. You may express your solutions in interval or in set notation.
   a) $|3x+4|<7$. Rewrite: $-7<3x+4<7$. $-11<3x<3$. $\boxed{-\frac{11}{3}<x<1}$
   b) $2x^2 \geq 5x$ $\Rightarrow 2x^2-5x\geq 0$, $x(2x-5)\geq 0$. Find endpoints $x=0$, $2x-5=0$, $2x=5$, $x=5/2$. + 0 − 5/2 + solution: $\boxed{\{x \mid x\leq 0 \text{ or } x\geq \frac{5}{2}\}}$
3. Find an equation of the line that satisfies the following conditions. Express your answers in slope-intercept form.
   a) Through the two points (-2, 3) and (1, -9) ① Find slope: $\frac{3-(-9)}{-2-1}=\frac{3+9}{-3}=\frac{12}{-3}=-4=m$ ② solve for b: $-9=-4(1)+b=-4+b$. $b=-5$; $\boxed{y=-4x-5}$ ← ③ write equation
   b) Through the point (5, 2) and parallel to the line $2x+5y=3$.
   ① find slope: $5y=-2x+3$; $y=-2/5x+3/5$, $m=-2/5$. ② Solve for b: $2=-2/5(5)+b=-2+b$. $b=4$. ③ Line: $\boxed{y=-\frac{2}{5}x+4}$
4. Find the distance between the points (-2, 3) and (1, -9) and the equation of the circle through the point (-2,3) with center at (1, -9).
   Distance $=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{(-2-1)^2+(3-(-9))^2}=\sqrt{(-3)^2+12^2}=\sqrt{9+144}=\boxed{\sqrt{153}}$
   $\boxed{(x-1)^2+(y+9)^2=(\sqrt{153})^2=153}$ (distance between the points is radius of the circle.)
5. Let $f(x)=\sqrt[3]{x+1}$ and $g(x)=e^{2x}$.
   a) What is the domain of $f(x)$? $\mathbb{R}$ (odd root - domain is all reals.)
   b) What is the domain of $g(x)$? $\mathbb{R}$ (exponential function - domain is all reals)
   c) Find the following functions and their domains:
      i) $f\circ g$ $=f(e^{2x})=\sqrt[3]{e^{2x}+1}$ Domain: $\mathbb{R}$
      ii) $g\circ f$ $=g(\sqrt[3]{x+1})=e^{2\sqrt[3]{x+1}}$ Domain: $\mathbb{R}$

6. Let $f(x)=\sqrt{\frac{x}{4}+2}$, $x \geq -8$.

a) Why is there the restriction on the domain of f? Because domain of even root functions is $x \geq 0$.

Let $\frac{x}{4}+2 \geq 0$; $\frac{x}{4} \geq -2$; $\boxed{x \geq -8}$

b) Find the function f^{-1} and its domain.

$y=\sqrt{\frac{x}{4}+2}$; $y^2=\frac{x}{4}+2$; $y^2-2=\frac{x}{4}$; $\boxed{4(y^2-2)=x=f^{-1}(y)}$. Domain: $y \geq 0$ (= range of $f(x)$).

c) How does the graph of $f(x)=\sqrt{\frac{x}{4}+2}$ compare to the graph of $g(x)=\sqrt{x}$ in terms of function transformations? left by 2 units, then stretched horizontally by a factor of 4.)

7. Let $g(x)=3+e^{2x}$.

a) What is the domain of g? $x \in \mathbb{R}$

b) What is the range of g? $y > 3$ (horizontal asymptote shifted up by 3.)

c) How does the graph of $g(x)=3+e^{2x}$ compare to the graph of $f(x)=e^{x}$ in terms of function transformations? Horizontal shrink by factor of ½, vertical shift up by 3 units.

d) Find the function inverse f^{-1}. Show that f and the function you found are indeed inverses by using function composition.

$y=3+e^{2x}$; $y-3=e^{2x}$; $\ln(y-3)=\ln e^{2x}=2x$; $\boxed{x=\frac{\ln(y-3)}{2}=f^{-1}(y)}$

$f \circ f^{-1}(y)=f\left(\frac{\ln(y-3)}{2}\right)=3+e^{\frac{2(\ln(y-3))}{2}}=3+e^{\ln(y-3)}=3+y-3=\boxed{y}$

$f^{-1} \circ f=f^{-1}(3+e^{2x})=\frac{\ln(3+e^{2x}-3)}{2}=\frac{\ln e^{2x}}{2}=\frac{2x}{2}=\boxed{x}$ ✓

8. Let $f(x)=\frac{4}{x}$.

a) Find the average rate of change between $x=1$ and $x=2$.

Avg rate of change $=\frac{f(2)-f(1)}{2-1}=\frac{\frac{4}{2}-\frac{4}{1}}{1}=\frac{2-4}{1}=\frac{-2}{1}=\boxed{-2}$

b) Find the average rate of change between $x=a$ and $x=a+h$.

Avg rate of change $=\frac{f(a+h)-f(a)}{a+h-a}=\frac{\frac{4}{a+h}-\frac{4}{a}}{h}=\frac{\frac{4a}{a(a+h)}-\frac{4(a+h)}{a(a+h)}}{h}=\frac{\frac{4a-4a-4h}{a(a+h)}}{h}=$

$\frac{\frac{-4h}{(a)(a+h)}}{h \cdot \frac{1}{h}}=\frac{-4h}{(a)(a+h)}\cdot\frac{1}{h}=\frac{\frac{-4h}{a(a+h)(h)}}{1}=\frac{-4}{(a)(a+h)}$

9. Let $f(x)=\begin{cases}-x^2+2, & if\ x \leq 0\\ x-1, & if\ x>0\end{cases}$

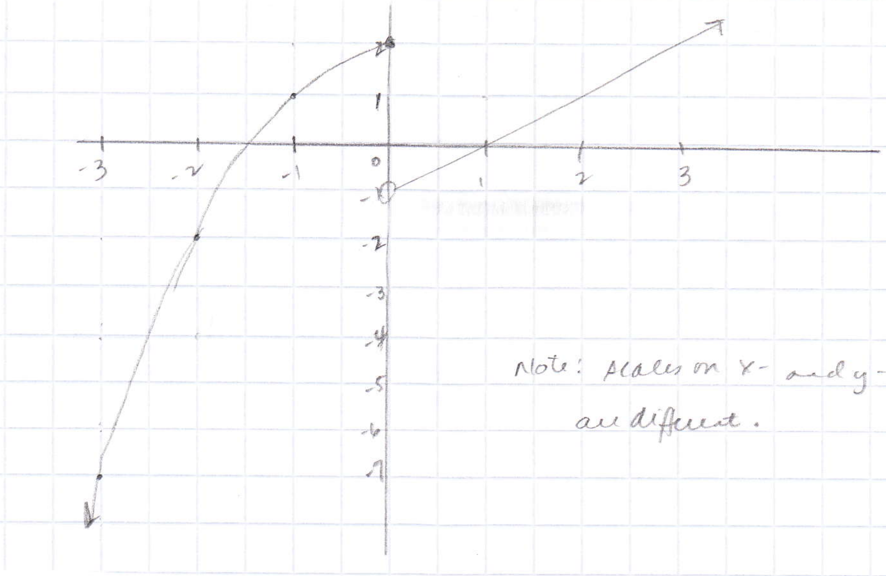
a) Find the following, showing all work:

i) $f(2)=2-1=1$

ii) $f(0)=-0^2+2=2$

iii) $f(-1)=(-1)^2+2=1+2=3$

b) Graph the piecewise defined function $f(x)$ on a domain of at least [-3...3]



10. Let $f(x) = 2x^2 - x$. Is f an even function, an odd function, or neither? Justify your answer.

Note: if even: $f(-x) = f(x)$. If odd: $f(-x) = -f(x)$.
1) Find $f(-x) = 2(-x)^2 - (-x) = 2x^2 + x$.
2) compare: $2x^2 + x \neq f(x)$. $\therefore$ not even
$-f(x) = -(2x^2 - x) = -2x^2 + x \neq 2x^2 + x$. $\therefore$ Not odd.
$\therefore f(x) = 2x^2 - x$ is neither even nor odd.

11. Find functions f and g such that the function $F(x) = \sqrt[5]{(x-4)}$ can be expressed in the form $f \circ g$.

Let $g(x) = x - 4$ $\quad f(x) = \sqrt[5]{x}$.
Then $f \circ g = f(g(x)) = f(x-4) = \sqrt[5]{x-4}$ ✓

12. Let $f(x) = x^2 - 8x + 10$.

a) Express $f(x)$ in standard form.

$b = -8$
$\frac{b}{2} = \frac{-8}{2} = -4$
$(\frac{b}{2})^2 = 16$

$f(x) = x^2 - 8x + 16 - 16 + 10$
$\boxed{= (x-4)^2 - 6 = f(x)}$

b) Find the vertex (both coordinates). Vertex: $(-h, k) = (4, -6)$

c) What is the y-intercept? $f(0) = 0^2 - 8(0) + 10 = 10 = y$ (or $(0, 10)$)

d) What is/are the x-intercepts, if any? Let $(x-4)^2 - 6 = 0$; $(x-4)^2 = 6$; $x - 4 = \pm\sqrt{6}$
$\boxed{x = 4 \pm \sqrt{6}}$

13. Consider the polynomial: $P(x) = x^3 + 2x^2 - 5x + 2$.

a) Use any method to show that $x = 1$ is a zero (root) of $P(x)$.

$P(1) = 1 + 2 - 5 + 2 = 0$. $\quad$ or

1	1	2	-5	2
		1	3	-2
	1	3	-2	0 ← (0 remainder)

May also do long division:

$$\begin{array}{r} x^2+3x-2 \\ x-1\overline{)\,x^3+2x^2-5x+2} \\ -(x^3-x^2) \qquad\qquad \\ \hline 3x^2-5x \quad \\ -(3x^2-3x) \quad \\ \hline -2x+2 \\ -(-2x+2) \\ \hline 0 \end{array}$$

$0 \leftarrow 0$ remainder

b) Find all zeros of the polynomial $P(x)$.

Let $x^2+3x-2=0$ Use quadratic formula: $x=\dfrac{-3\pm\sqrt{3^2-4(1)(-2)}}{2\cdot 1}=\dfrac{-3\pm\sqrt{9+8}}{2}=\dfrac{-3\pm\sqrt{17}}{2}$

$\boxed{\text{Zeros are } x=1,\ x=\dfrac{-3+\sqrt{17}}{2},\ x=\dfrac{-3-\sqrt{17}}{2}}$

14. Let $P(x)=x^4-x^3-6x^2$

a) Find all real zeros of $P(x)$. $x=0,\ x=3,\ x=-2$

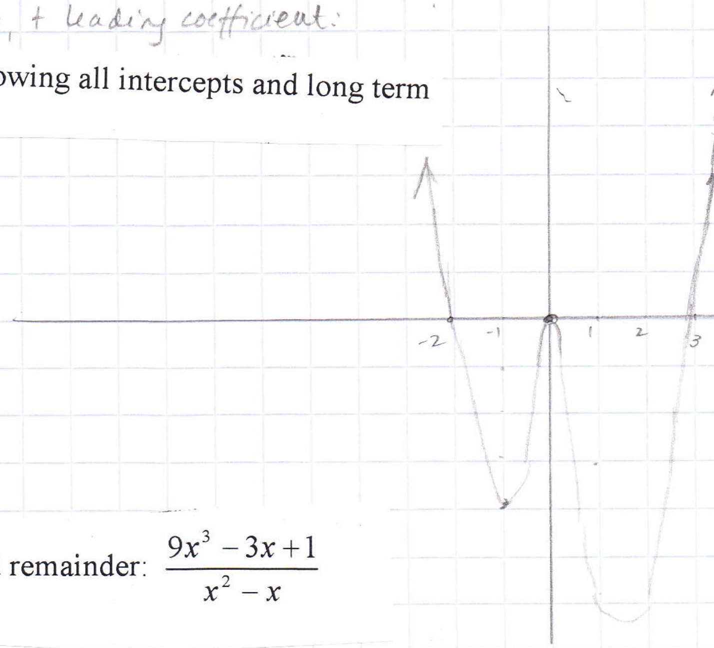
$P(x)=x^4-x^3-6x^2=x^2(x^2-x-6)=x^2(x-3)(x+2)$

Let $x^2(x-3)(x+2)=0$

$x=0 \quad x=3 \quad x=-2$ (double root at $x=0$)

Even degree, + leading coefficient:

b) Sketch the graph of $P(x)$, accurately showing all intercepts and long term behavior.



x intercepts: $x=0, 3, -2$

y-intercept: $y=0$

Values

x	$f(x)$
-3	$81+27-54=54$
-1	$1+1-6=-$
1	$1-1-6=-6$
4	$256-64-96=90$

15. Find the quotient and remainder: $\dfrac{9x^3-3x+1}{x^2-x}$

$$\begin{array}{r} 9x+9 \\ x^2-x\overline{)\,9x^3+0x^2-3x+1} \\ -(9x^3-9x^2) \qquad\quad \\ \hline 9x^2-3x \quad \\ -(9x^2-9x) \quad \\ \hline 6x+1 \end{array}$$

$9x+9 \leftarrow$ Quotient

$6x+1 =$ Remainder

OR: $\dfrac{9x^3-3x+1}{x^2-x}=\boxed{9x+9+\dfrac{6x+1}{x^2-x}}$

16. Let $g(x)=\dfrac{12+3x^2}{x^2-2x-3}$. Find the following:

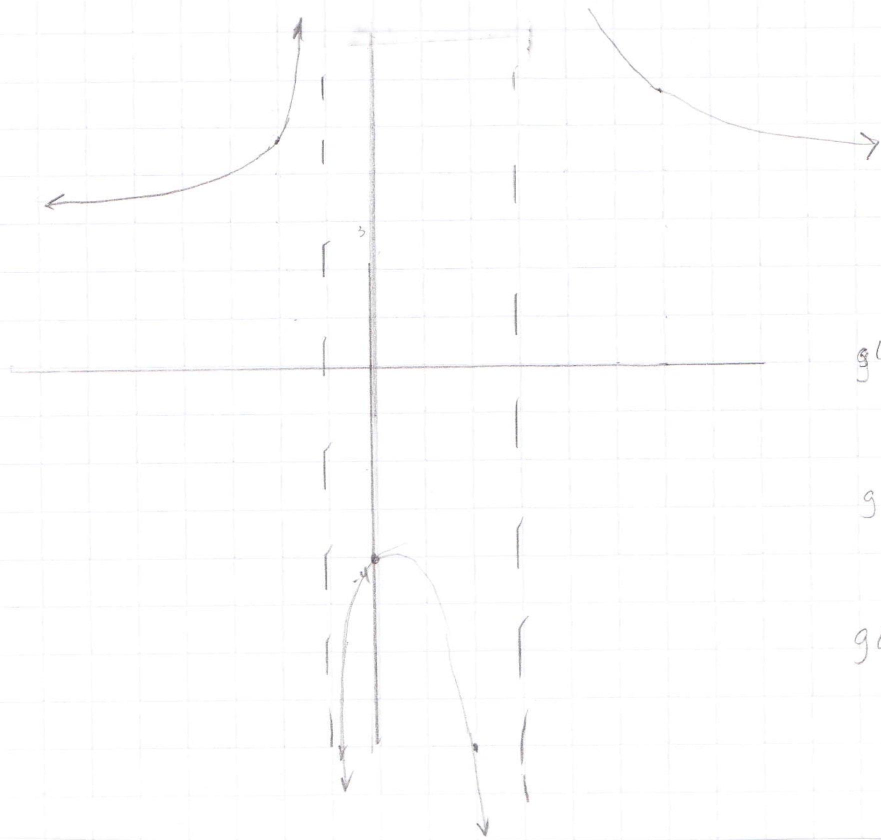
a) x-intercepts, if any None   Let $12+3x^2=0$; $12=-3x^2$ $-4=x^2$ No real solutions

b) y-intercept, if any $y=-4$, or $(0,-4)$ $g(0)=\frac{12+0}{0-3}=-4$

c) Vertical asymptote(s), if any $x=3$, $x=-1$ Let $x^2-2x-3=0$; $(x-3)(x+1)=0$, $x=3$, $x=-1$

d) Horizontal asymptote, if any $y=3$ $g(x)=\frac{3x^2+12}{x^2-2x-3}$, Ratio of leading coefficient $=3$.

e) Graph the function $g(x)=\dfrac{12+3x^2}{x^2-2x-3}$



$g(2)=\frac{12+3(4)}{4-4-3}=\frac{24}{-3}=-8$

$g(4)=\frac{12+48}{16-8-3}=\frac{60}{5}=12$

$g(6)=\frac{12+3(36)}{36-12-3}=\frac{120}{21}\approx 5.7$

$g(-2)=\frac{12+12}{4+4-3}=\frac{24}{5}=4\frac{4}{5}$

17. Evaluate the following expressions, eliminating the logarithms. (Your answer in each case should be a number.)

a) $\log 20+\log 5=\log(20\cdot 5)=\log 100=\log 10^2=\boxed{2}$

b) $4^{2\log_4 5}=4^{\log_4 5^2}=4^{\log_4 25}=\boxed{25}$

c) $6\ln\sqrt[3]{e}=6\ln e^{1/2}=\ln(e^{1/3})^6=\ln e^2=\boxed{2}$ (or $6\ln e^{1/3}=6\cdot\frac{1}{3}=2$)

18. Rewrite the expression as a single logarithm: $\ln(x+4)+\ln(x-4)-4\ln x=$ $\ln(x+4)-\ln(x-4)-\ln x^4=$

$\ln\left[\frac{(x+4)(x-4)}{x^4}\right]$

19. Solve for x.

a) $5^{3x-7}-7=18$

$+7\quad +7$

$5^{3x-7}=25=5^2 \Rightarrow 3x-7=2$

$3x=9$

$\boxed{x=3}$

Rewrite exponentially:

b) $1 = \log 4 - \log(x+1)$ $= \log \frac{4}{x+1}$; $10^1 = \frac{4}{x+1} \Rightarrow x+1 = \frac{4}{10}$; $x = \frac{4}{10} - 1 = -\frac{6}{10} = \frac{-3}{5}$

c) $\ln x^2 = \ln 2 + \ln x$ $\ln x^2 = \ln 2x$; $x^2 = 2x$; $x^2 - 2x = 0 = (x)(x-2)$

$\cancel{x=0}$; $\boxed{x=2}$ Check: $x=0$ not in domain

20. True/False. Determine whether each of the following statements is true or false. Show work to justify your answer.

a) F $e^{\ln x - \ln y} = x - y$ $e^{\ln x - \ln y} = e^{\ln \frac{x}{y}} = \frac{x}{y} \neq x - y$

b) T $\log(2000x) = 3 + \log x + \log 2$. $\log(2000x) = \log(1000 \cdot 2 \cdot x) = \log 1000 + \log 2 + \log x = \log 10^3 + \log 2 + \log x = 3 + \log 2 + \log x$ ✓

c) T If $\log y = 3 + 2x$ then $y = (1000)\cdot(100^x)$ $\log y = 3 + 2x \Rightarrow 10^{3+2x} = 10^3 \cdot 10^{2x} = 1000 \cdot (10^2)^x = 1000 \cdot 100^x$ ✓

d) F If $y = e^x \cdot e^2$, then $\ln y = 2x$. $y = e^x \cdot e^2 \Rightarrow y = e^{x+2}$; $\ln y = \ln(e^{x+2}) = x + 2 \neq 2x$

21. Consider the angle $-\frac{5\pi}{4}$.

a) Express $-\frac{5\pi}{4}$ radians as degrees: $-225°$ $-\frac{5\pi}{4} \cdot \frac{180°}{\pi} = -5(45) = -225°$

b) What is the reference angle for this angle (in radians)? $\pi/4$ radians

c) In what quadrant does the terminal side of this angle lie? II

d) What is the exact value of $\sin -\frac{5\pi}{4}$? $\frac{\sqrt{2}}{2}$

What is the exact value of $\tan -\frac{5\pi}{4}$? -1

22. Consider the angle 250°.

a) Express 250° as radians: $\frac{25\pi}{18}$ $250° \cdot \frac{\pi}{180°} = \frac{250\pi}{180} = \frac{25\pi}{18}$

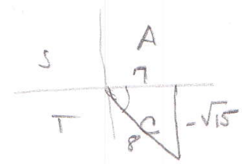
b) What is the reference angle for this angle (in radians or degrees)? $70° = \frac{7}{18}\pi$ $(250° - 180°)$

c) In what quadrant does the terminal side of this angle lie? III

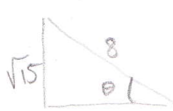
$\cos\theta > 0$
$\sin\theta < 0$
$\Rightarrow$ Quadrant IV

23. Find the exact values of the following functions of the angle θ, when $\cos\theta = \frac{7}{8}$, and $\sin\theta < 0$. First figure out in which quadrant the terminal side of θ must lie:

Quadrant IV. a) $\sin\theta =$ $-\sqrt{15}/8$; b) $\tan\theta =$ $-\sqrt{15}/7$;



c) $\sec\theta =$ $8/7$; d) $\csc\theta =$ $-\frac{8}{\sqrt{15}}$; e) $\cot\theta =$ $-\frac{7}{\sqrt{15}}$



$7^2 + b^2 = 8^2$
$49 + b^2 = 64$
$b^2 = 64 - 49 = 15$
$b = \sqrt{15}$

24. Find the exact value for the following: $-\frac{14\pi}{3}$ is coterminal with $-2\pi/3$

a) $\cos -\frac{14\pi}{3} =$ $-1/2$

Q III. Ref ∠ $= \pi/3$

b) $\tan(135°) =$ -1

Q II, ref ∠ $= 45° = \pi/4$ ($\tan\theta$ is negative in Q II)

25. Evaluate the expressions: $\sin^2(-150°) + \cos^2(210°) =$ and $4\tan^2\frac{\pi}{5} - 4\sec^2\frac{\pi}{5}$.

① Note: $-150°$ & $210°$ are coterminal, so $\sin^2(-150°) + \cos^2(210°) = \boxed{1}$

② Relationship: $\tan^2\theta + 1 = \sec^2\theta$. So $\tan^2\theta - \sec^2\theta = -1$. $4(\tan^2\frac{\pi}{4} - \sec^2\pi/4) = 4(1) = \boxed{4}$

26. Find the radius of the circle if an arc of length 15 m on the circle subtends a central angle of 30°.(Hint: $S = r\theta$) $90/\pi$ meters

① convert 30° to radians: $30° \cdot \frac{\pi}{180°} = \frac{\pi}{6}$ radians

$15m = r \cdot \frac{\pi}{6}$; $r = \frac{15 \cdot 6}{\pi} = \frac{90}{\pi}$ meters