The Laplacian with General Robin Boundary Conditions

DISSERTATION
zur Erlangung des Doktorgrades Dr. rer. nat.
der Fakultät für Mathematik und Wirtschaftswissenschaften
der Universität Ulm

vorgelegt von
Mahamadi Warma
aus Burkina Faso

Juni 2002

Dekan	Prof. Dr. D. Beschorner	(Ulm, Germany)
1. Gutachter	Prof. Dr. W. Arendt	(Ulm, Germany)
2. Gutachter	Prof. Dr .F. Schulz	(Ulm, Germany)
3. Gutachter	Prof. Dr. D. Daners	(University of Sydney)

Tag der mündlichen Prüfung: 04. Juli 2002

A mes parents qui m'ont quitté très tôt

Contents

Introduction

The analysis of mathematical models for physical phenomena is part of the subject matter of mathematical physics. In all cases the mathematical problems which arise lead to more general mathematical questions not associated with any particular model. Although these general questions are sometimes problems in pure mathematics, they are usually classified as mathematical physics since they arise from problems in physics.

Mathematical physics has traditionally been concerned with the mathematics of classical physics, mechanics, fluid dynamics, acoustics, potential theory and optics. The main mathematical tool for the study of these branches of physics is the theory of ordinary and partial differential equations and related areas like integral equations and the calculus of variations.

But in the theory of partial differential equations, an important problem is the question concerning the existence of solutions when the values on the boundary of the region are prescribed. "Has not every variational problem a solution, provided certain assumptions regarding given boundary conditions are satisfied and provided also, if need be, that the notion of solution shall be suitably extended"? These are the words of Hilbert used to conclude his question concerning the twentieth problem stated in an address delivered before the International Congress of Mathematicians in 1900.

Although there are still many open questions related to these problems of Hilbert, a great deal of progress has been made, with some dazzling success.

One of the components necessary to establish regularity of certain variational problems was the need to show that a weak solution of a linear (or nonlinear) equation in divergence form is regular in some sense.

Potential theory, which grew out of the theory of the electrostatic or gravitational potential, the Laplace equation, the Dirichlet problem, ..., played a fundamental role in the development of functional analysis and the theory of Hilbert space. The connection between potential theory and the theory of Hilbert spaces can be traced back to Gauss, who proved the existence of equilibrium potentials by minimizing a quadratic integral, the energy. According to the classical Dirichlet principle, one obtains the solution of Dirichlet's problem for the Laplace equation in a region Ω by minimizing the Dirichlet integral, $\int_\Omega |\nabla u(x)|^2 \, dx$, over a certain class of functions taking given values on the boundary $\partial\Omega$. The natural explanation is that solutions of Laplace equation describe an equilibrium state, a state attained when the energy carried by the system is at a minimum.

It turns out that potential theory is the main tool for the study of regularity of weak solutions of linear (or nonlinear) equations in divergence form. For example, the inspired result by Wiener characterizes continuity at the boundary for harmonic functions. Serrin discovered that capacity was the appropriate measurement for describing removable sets for weak solutions. Later, Maz'ya discovered a Wiener-type expression involving capacity which provided a sufficient condition for continuity at the boundary of weak solutions of equations whose structure is similar to that of the p-Laplacian.

In particular, potential theory for Dirichlet forms, mainly due to Beurling and Deny, has many probabilistic interpretations and is also connected with the classical potential theroy based on Riesz kernels and logarithmic kernels, see [2], [55] or [73]. This theory of Dirichlet forms is an axiomatic extension of the classical Dirichlet integrals, see for example [2], [55], [69] or [84].

We will be concerned with Dirichlet forms associated with some realizations of the Laplacian on $L^2(\Omega)$ where Ω is a region in $\mathbb{R}^N$. It turns out that Sobolev spaces which have very interesting mathematical structures in their own right, will play an important role here. The associated realizations of the Laplacian have some properties which are consequences of the structure of Sobolev spaces which themselves are related to the structure of the geometry of Ω.

Since open subsets of $\mathbb{R}^N$ may have a strange geometry, the method of quadratic forms is the main tool to define realizations of the Laplacian with various boundary conditions. For an arbitrary open set there is no problem to define in the weak sense the Dirichlet Laplacian Δ_D and the Neumann Laplacian Δ_N on $L^2(\Omega)$ and it is well-known that these operators generate holomorphic contractive C_0-semigroups which interpolate on $L^p(\Omega)$ for $1 \leq p < \infty$ (see [8], [16] or [39]).

Before 1998, the third type of boundary conditions called Robin boundary conditions has been considered only in the case of regular open sets (for example Lipschitz domains), see [16], [38] or [59]. The difficulty is to find an appropriate measure on the boundary and the fact that there may exist functions in the first order Sobolev space $H^1(\Omega)$ which have no trace in some appropriate Hilbert space if Ω is "bad".

Daners [34] found a way to give a weak formulation of Robin boundary conditions on arbitrary bounded domains. First, he chose as measure the $(N-1)$-dimensional Hausdorff measure restricted to $\partial\Omega$ which seemed to be the natural candidate since this measure coincides with the usual Lebesgue surface measure if Ω has a Lipschitz boundary. After the choice of the measure, he proved that there exists a natural subset S of $\partial\Omega$ where Robin boundary conditions are realized and one has Dirichlet boundary conditions on $\partial\Omega \setminus S$. He conjectured that S is always equal to $\partial\Omega$ if $\partial\Omega$ has finite $(N-1)$-dimensional Hausdorff measure.

In this thesis we will be concerned with Robin boundary conditions, not only on bounded domains, but on arbitrary open sets.

The first part of this thesis is organized as follows.

In Chapter 2, we will introduce a new notion of capacity called relative capacity (relative to Ω) and study the relationship between this relative capacity and the well-known classical capacity considered by many authors (see [2], [15], [19] [23], [43], [55] or [73]). It will be of particular interest to compare the relative capacity and the s-dimensional Hausdorff measure for $N-1 \leq s \leq N$. It is well-konwn (see [2], [19] or [43]) that sets of zero capacity have also zero s-dimensional Hausdorff measure. We will show by several examples that this statement is not true for the relative capacity.

In Chapter 3 we will consider the bilinear symmetric form on $L^2(\Omega)$ defined by

$$a_\mu(u,v) = \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} uv \, d\mu$$

with domain

$$D(a_\mu) = \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \int_{\partial\Omega} |u|^2 \, d\mu < \infty\}$$

where μ is a Borel measure on $\partial\Omega$. If the form a_μ is closable then we can associate with its closure a selfadjoint realization Δ_μ of the Laplacian on $L^2(\Omega)$. The operator Δ_μ is called the Laplacian with general Robin boundary conditions.

Therefore it is important to know when a_μ is closable. This is similar to the study of an abstract perturbation of Dirichlet forms by measures considered by Fukushima, Oshima and Takeda in [55], Ma and Röckner in [69] or Stollmann and Voigt in [89]. It turns out that a_μ is closable if and only if μ does not charge Borel subsets of zero relative capacity of the part on which it is locally finite.

If a_μ is not closable, the remarkable result of Reed and Simon [84] shows that there exists a largest closable part smaller than a_μ in some sense. Using the relative capacity approach we will show that this largest closable part is in fact a_{μ_r} where μ_r is the restriction of μ to some maximal Borel subset S of $\partial\Omega$. The selfadjoint operator Δ_μ on $L^2(\Omega)$ associated with the closure of this closable part satisfies general Robin boundary conditions on S and Dirichlet boundary conditions on $\partial\Omega \setminus S$.

In both cases, Δ_μ generates a holomorphic submarkovian C_0-semigroup on $L^2(\Omega)$ which is sandwiched between the Dirichlet Laplacian and the Neumann Laplacian semigroups. Conversely, we will also show that under a locality and a regurality assumption, each sandwiched semigroup is given by a measure on $\partial\Omega$ which does not charge Borel subsets of zero relative capacity of the part on which it is locally finite.

Chapter 4 will concern to the study of the Laplacian with classical Robin boundary conditions and of some properties of the Neumann Laplacian. The classical Robin boundary condition corresponds to the case where $\mu = \sigma$ is the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure or more generally

μ absolutely continuous with respect to σ. For simplicity we will consider only the case $\mu = \sigma$. We will show by examples that Daners' conjecture mentioned above is not true. This follows by the geometry of Ω. If Ω has a very bad boundary it can happen that σ charges subsets of $\partial\Omega$ of zero relative capacity. We will illustrate this situation by several examples. Maz'ya [73] shows that for an arbitary open set in $\mathbb{R}^N$ the space $W^1_{2,2}(\Omega, \partial\Omega)$ is continuously embedded into $L^{2N/N-1}(\Omega)$. The non-closability of a_σ implies in particular that the continuous embedding of $W^1_{2,2}(\Omega, \partial\Omega)$ into $L^{2N/N-1}(\Omega)$ is not always injective. However, the Laplacian with classical Robin boundary conditions has some interesting properties (p-independence of the spectrum, Gaussian estimates with modified exponent of the associated semigroup, compactness of the resolvent on $L^p(\Omega)$, $1 \leq p \leq \infty$ if Ω has finite measure) which are direct consequences of the remarkable Maz'ya inequality.

The Neumann Laplacian corresponds to the case where $\mu = 0$. This operator has some very strange properties (the spectrum may be p-dependent, the resolvent is not always compact if Ω is bounded). However, we will show that the associated semigroup is always an integral operator, but it is given by a singular kernel; i.e. a kernel which is not bounded if Ω is irregular.

The second part of this thesis is concerned with the study of regularity of weak solutions and the introduction of the fourth boundary conditions called Wentzell-Robin boundary conditions.

As mentioned above, one of the component necessary to establish regularity of certain variational problems is the need to show that a weak solution of a linear equation in divergence form with bounded measurable coefficients is Hölder continuous. This result resisted many attempts, but finally in 1957, De Giorgi and Nash, independently of each other, provided a proof of this crucial result. The De Giorgi-Nash result stimulated a great number of related investigations, one of most important being that of Moser who, by an entirely different method, provided another proof of their result. A crucial component in Moser's proof was the discovery that the logarithm of the solution is a function of bounded mean oscillation. He also proved the Harnack inequality which states that locally, the supremum of the solution is bounded by its infimum.

In Chapter 5, we will consider the inhomogeneous Robin problem given formally by

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} + \beta u = g & \text{on} \quad \partial\Omega \end{cases}$$

where $f \in L^p(\Omega)$ $(p \geq 2)$, $g \in L^q(\partial\Omega)$ $(q \geq 2)$ and β is a strictly positive bounded measurable function on $\partial\Omega$. Using the De Giorgi method developped by Murthy and Stampacchia [77], we will show that a weak solution of the inhomogeneous Robin problem is Hölder continuous up to the boundary provided that $p > N$, $q > N - 1$ and Ω is a bounded domain with Lipschitz boundary. This shows in particular that the operator Δ_σ defined in Chapter 4 has the strong Feller property. Furthermore, we show that the part of Δ_σ in $C(\bar{\Omega})$ generates a holomorphic contractive C_0-semigroup on $C(\bar{\Omega})$.

Most recently, Favini et al. [48] investigated the Laplacian with Wentzell-Robin boundary conditions. They considered L^p-spaces for bounded domains in $\mathbb{R}^N$ but investigated explicitly the interval $[0,1]$. They proved that the Wentzell-Robin Laplacian generates a holomorphic C_0-semigroup on some L^p-spaces. Moreover it generates a C_0-semigroup on $C(\bar{\Omega})$ if Ω is regular (for example if Ω is of class C^2). In Chapter 6, we will prove that the C_0-semigroup on $C[0,1]$ generated by the Wentzell-Robin Laplacian is also holomorphic .

Acknowledgement

I have the pleasure to thank at least some of the people and organizations, without whom this thesis would not have viewed the day.

First of all, I would like to thank my advisor Wolfgang Arendt. Too much of this thesis and the last few years depended on his support and guidance. I also thank him for giving me the posssibility to learn german and to study at the university of Ulm. In french, one says: “A lui vont tous mes remerciements”.

Next, I would like to thank Ralph Chill for his hospitality, his disponibility to answer any question (mathematics or life). I am grateful to him for reading this thesis and for his constructive remarks. Let me just say that to me he is an exceptional friend.

During the last year, I had some interesting mathematical discussions with Markus Biegert. I am grateful to him and I encourage him to understand the mystery of Sobolev spaces.

I also want to thank all members of the Abteilung Angewandte Analysis for the stimulating working atmosphere and for diverse help. Special thanks go to Markus Duelli who, not only, did not mind sharing a room with me, but also for some interesting discussions on subjects like mathematics or football. Of course, I also do not want to forget to thank Markus Haase, Sonja Thomaschewski and Stefan Karrmann for diverse help.

Next, it is a pleasure to thank Prof. V. G. Maz'ya for receiving me two times, and for many helpful discussions and for his orientation to some new perspective.

I also want to thank Professor D. Daners for many helpful discussions.

Next, I would like to thank Prof. F. Schulz for his help.

I would like to thank several people who encouraged me during my study. First of them is H. Touré who introduced me to Wolfgang Arendt. My special thanks go to V. Ambeu, I. Guené, S. Niyndagiye and to N. J. N. Koussoubé who stimulated me during the last year.

To finish, I would like to thank the DAAD for giving me the possibility to come to Germany. This thesis was partially financially supported by the DFG “Regularität und Asymptotik für elliptische und parabolische Probleme”.

Chapter 1

Basic Notions of Functional Analysis

This chapter contains some basic notions which will facilitate the understanding of the following chapters. We will not give a proof of most of the results. We simply give some references where one can find these results.

1.1 Measure Theory.

Throughout this section, X will denote a metric space with metric d and $\mathcal{B}(X)$ the **Borel σ-algebra** of subsets of X; i.e., the smallest σ-algebra containing the open subsets of X. The following presentation is based on the notions and results of Evans-Gariepy [43].

Definition 1.1.1. *a) A mapping $\mu : [A : A \subset X] \to [0, \infty]$ is called an* **outer measure** *on X if the following two conditions are satisfied.*

(i) $\mu(\emptyset) = 0$*;*

(ii) $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$ *whenever $A_i \subset X$.*

*b) A set $A \subset X$ is μ-***measurable** *if for each set $B \subset X$,*

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A).$$

Remark 1.1.2. *If μ is an outer measure on X and $A \subset B \subset X$, then $\mu(A) \leq \mu(B)$.*

Definition 1.1.3. *Let μ be an outer measure on X.*

a) We say that μ is **locally finite** *if for every $x \in X$ there exists $r > 0$ such that $\mu(B(x, r)) < \infty$, where $B(x, r) := \{y \in X : d(x, y) < r\}$.*

b) *We say that μ is a* **Borel measure** *if all Borel sets are μ-measurable.*

c) *We say that μ is a* **regular Borel measure** *if it is a Borel measure and if for every $A \subset X$ there exists a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.*

d) *We say that μ is a* **Radon measure** *if it is a Borel measure and*

(i) *$\mu(K) < \infty$ for compact sets $K \subset X$;*

(ii) *$\mu(O) = \sup\{\mu(K) : \ K \subset O \ \text{ is compact } \}$ for open sets $O \subset X$.*

(iii) *$\mu(A) = \inf\{\mu(O) : \ A \subset O; \ O \ \text{ is open } \}$ for $A \subset X$.*

Remark 1.1.4. a) *It follows from the definition that an outer measure μ on X is a Radon measure if and only if it is a regular Borel measure and locally finite.*

b) *Let μ be a regular Borel measure on X. If $A \subset X$ is a Borel set, then μ* **restricted** *to A denoted by $\mu|_A$ and defined by*

$$\mu|_A(B) := \mu(A \cap B) \ \text{ for all } \ B \subset X$$

is a regular Borel measure.

Example 1.1.5 (Hausdorff measure). Let U be a nonempty subset of $\mathbb{R}^N$. We define the diameter of U as follows:

$$\operatorname{diam}(U) := \sup\left\{|x-y| : \ x, y \in U\right\}.$$

Let $0 < \delta \leq \infty$ and $F \subset \mathbb{R}^N$. If $\{U_i\}$ is a countable family of subsets of $\mathbb{R}^N$ such that $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 < \operatorname{diam}(U_i) \leq \delta$ for each i, then $\{U_i\}$ is called a **δ-covering** of F. Let $F \subset \mathbb{R}^N$, $0 \leq s < \infty$ and $0 < \delta \leq \infty$. Define

$$\mathcal{H}^s_\delta(F) := 2^{-s}\alpha(s) \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam}(U_i))^s : \ \{U_i\} \ \text{ is a } \ \delta\text{-covering of } \ F\right\}$$

where $\alpha(s) := \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$ and $\Gamma(s) := \int_0^\infty e^{-x}x^{s-1}\,dx$ is the usual **gamma function**. Since $\mathcal{H}^s_\delta(F) \leq \mathcal{H}^s_\varepsilon(F)$ if $0 < \varepsilon < \delta \leq \infty$, it follows that $\mathcal{H}^s_\delta(F)$ approaches a limit as $\delta \to 0$. Define

$$\mathcal{H}^s(F) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(F) = \sup_{\delta > 0} \mathcal{H}^s_\delta(F).$$

The limit exists for every $F \subset \mathbb{R}^N$ and can be 0 or ∞. We call $\mathcal{H}^s$ the **s-dimensional Hausdorff measure**.

For each subset F of $\mathbb{R}^N$ and $0 \leq s < \infty$, by definition of $\mathcal{H}^s(F)$, we see that if $\mathcal{H}^s(F) < \infty$ then $\mathcal{H}^t(F) = 0$ for $t > s$. Thus a graph of $\mathcal{H}^s(F)$ shows that there is a critical value of s at which $\mathcal{H}^s(F)$ “jumps” from ∞ to 0. This critical value is called the **Hausdorff dimension** of F and written $\dim_H(F)$. Clearly,

$$\dim_H(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$

so that

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } s < \dim_H(F) \\ 0 & \text{if } s > \dim_H(F). \end{cases}$$

If $s = \dim_H(F)$, then $\mathcal{H}^s(F)$ may be zero of infinite, or may satisfy

$$0 < \mathcal{H}^s(F) < \infty.$$

Let $\Omega \subset \mathbb{R}^N$ be an open set and $\partial\Omega$ be its boundary. Then

$$(N-1) \leq \dim_H(\partial\Omega) \leq N.$$

The following result is contained in [46, Section 2.1], [72, Theorem 4.2 and Corollary 4.5] and [4, Proposition 2.49].

Theorem 1.1.6. *a) The s-dimensional Hausdorff measure on $\mathbb{R}^N$ is a regular Borel measure but not always a Radon measure.*

b) The measure $\mathcal{H}^N$ coincides with the Lebesgue measure on $\mathbb{R}^N$.

Next we give the following formula called **area formula** and contained in [4, Theorem 2.71]

Theorem 1.1.7 (Area formula). *Let $f : \mathbb{R}^k \to \mathbb{R}^N$ be a* **Lipschitz function** *with $N \geq k$. Then for every Lebesgue measurable set $E \subset \mathbb{R}^k$ the* **multiplicity function** *$y \mapsto \mathcal{H}^0(E \cap f^{-1}(y))$ is $\mathcal{H}^k$-measurable on $\mathbb{R}^N$ and*

$$\int_{\mathbb{R}^N} \mathcal{H}^0(E \cap f^{-1}(y))\, d\mathcal{H}^k(y) = \int_E Jf(x)\, dx,$$

where Jf is the Jacobian of f.

Remark 1.1.8. *a) The set $f(E)$ is $\mathcal{H}^k$-measurable, being the support of the multiplicity function. If f is injective, then*

$$\mathcal{H}^k(f(E)) = \int_E Jf(x)\, dx.$$

b) Assume that $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is Lipschitz and define $f : \mathbb{R}^{N-1} \to \mathbb{R}^N$ by $f(x) = (x, g(x))$. Calculating, we obtain that $(Jf)^2 = 1 + |Dg|^2$. For each open set $U \subset \mathbb{R}^{N-1}$, define the graph of g over U,

$$G := G(g, U) := \{(x, g(x)) : x \in U\} \subset \mathbb{R}^N.$$

Then

$$\mathcal{H}^{N-1}(G) = S(G) = \int_U (1 + |Dg|^2)^{1/2}\, dx.$$

c) *It follows from b) that if* $\Omega \subset \mathbb{R}^N$ *is an open set with* **Lipschitz boundary**; *i.e., the boundary is locally the graph of a Lipschitz function, then the restriction to* $\partial\Omega$ *of the* $(N-1)$*-dimensional Hausdorff measure* $\mathcal{H}^{N-1}$ *coincides with the usual Lebesgue surface measure on* $\partial\Omega$.

1.2 Banach Lattices.

For more information on the theory of Banach lattices, we refer to the monograph Schaefer [86].

Definition 1.2.1. *An ordered vector space* E *is called a* **vector lattice** *if any two elements* $f, g \in E$ *have a supremum, which is denoted by* $f \vee g$, *and an infimum, denoted by* $f \wedge g$.

Remark 1.2.2. a) $f \vee (-f) = |f|$ *is called the* **absolute value** *of* f.

b) $f \vee 0 = f^+$ *is called the* **positive part** *of* f.

c) $-(f \wedge 0) = f^-$ *is called the* **negative part** *of* f.

Let E be a vector lattice. One of the following equivalent assertions is a necessary and sufficient condition on a vector subspace G of E to be a **vector sublattice**.

(i) $h \in G \Rightarrow |h| \in G$.

(ii) $h \in G \Rightarrow h^+ \in G$.

(iii) $h \in G \Rightarrow h^- \in G$.

Definition 1.2.3. *Let* E *be a vector lattice.*

a) *A linear subspace* I *of* E *is called an* **ideal** *if* $f \in I$ *and* $g \in E$ *such that* $|g| \le |f|$ *imply* $g \in I$.

b) *A subspace* B *of* E *is a* **band** *if* B *is an ideal of* E *and* $\sup M$ *is contained in* B *whenever* M *is contained in* B *and has a supremum in* E.

c) *A norm on* E *is called a* **lattice norm** *if it satisfies*

$$|f| \le |g| \Longrightarrow \|f\| \le \|g\|. \tag{1.1}$$

d) *A* **Banach lattice** *is a Banach space* E *endowed with an ordering* $\le$ *such that* $(E, \le)$ *is a vector lattice and the norm on* E *is a lattice norm.*

The following result due to Schaefer (see [86, Example 2 p.157-158]) characterizes the closed ideals in the Banach lattice $L^p(X,\mu)$ for some **σ-finite measure** μ on X; i.e., we can write $X = \bigcup_{i=1}^{\infty} X_i$ where X_i is μ-measurable and $\mu(X_i) < \infty$ for $i = 1, 2, \ldots$.

Theorem 1.2.4. *Let μ be a σ-finite measure on X and $1 \leq p < \infty$. Let I be a closed ideal of $L^p(X,\mu)$. Then there exists a measurable subset S of X such that*

$$I = \{f \in L^p(X,\mu) : \; f = 0 \; \mu\text{-a.e. on } S\}.$$

1.3 Semigroups Generated by Dirichlet Forms.

Throughout this section, B will denote a Banach space.

Definition 1.3.1. *We call* **C_0-semigroup** *on B any family $T = (T(t))_{t\geq 0}$ of bounded operators on B such that:*

a) $T(t+s) = T(t)T(s) \quad t,s \in \mathbb{R}_+$

b) $T(0) = I$ *identity on* B

c) $T(t)x \to x$ *as* $t \downarrow 0$ *for all* $x \in B$.

The **generator** *A of the semigroup $T = (T(t))_{t\geq 0}$ is the operator defined on the domain*

$$D(A) = \left\{ x \in B : \; \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \;\; \text{exists in } \; B \right\}$$

by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \;\; \text{for all } \; x \in D(A).$$

The notion of semigroups has been introduced for the study of the evolution equation

$$(PC) \quad \begin{cases} u'(t) = & Au(t), \;\; t \geq 0 \\ u(0) = & x \end{cases}$$

called **Cauchy problem**

Definition 1.3.2. *Let $\theta \in (0, \frac{\pi}{2}]$. A semigroup T on B is called* **holomorphic** *of angle θ if it has a holomorphic extension to $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : \; |\arg z| < \theta\}$ such that*

$$\lim_{z\to 0} T(z)x = x \quad \text{for all } \; x \in B.$$

In this case, $T(z+z') = T(z)T(z')$ for all $z, z' \in \Sigma_\theta$.

In this section, we will be concerned with semigroups given by Dirichlet forms which we will define below. In our definition of forms, we shall consider only symmetric forms. For a general theory, we refer to [65], [69] and [79] where non-symmetric forms have been considered, too.

Let H be a fixed Hilbert space and $D(a) \hookrightarrow H$. Let $a : D(a) \times D(a) \to \mathbb{R}$ be a bilinear positive symmetric form. For $u, v \in D(a)$ we let

$$\|u\|_a^2 := a(u,u) + \|u\|_H^2 \;\; \text{and} \;\; a_1(u,v) := a(u,v) + (u,v)_H$$

where $(,)_H$ denotes the scalar product on H. The space $D(a)$ is called the **domain** of a.

Definition 1.3.3. *The form $(a, D(a))$ is said to be* **closed** *on H if $(D(a), \|\cdot\|_a)$ is a Hilbert space. More precisely, the symmetric form $(a, D(a))$ is said to be closed on H if*

$$u_n \in D(a),\ a_1(u_n - u_m, u_n - u_m) \to 0 \quad as \quad n, m \to \infty$$

$$\Rightarrow\ \exists\, u \in D(a):\ a_1(u_n - u, u_n - u) \to 0 \quad as \quad n \to \infty.$$

Let $(a, D(a))$ be a closed symmetric form on H. Then we can define a selfadjoint operator A on H associated with a in the following way:

$$\begin{cases} D(A) := \{u \in D(a):\ \exists\, v \in H:\ (v, \varphi)_H = a(u, \varphi)\ \forall\, \varphi\ \in D(a)\} \\ Au := -v. \end{cases}$$

The proof of the following result is contained in [35, Chap. XVII p.450].

Theorem 1.3.4. *Assume that $D(a)$ is dense in H. Then the operator A generates a holomorphic C_0-semigroup $T = (e^{tA})_{t \geq 0}$ on H.*

Next we give examples of closed forms. Let $\Omega \subset \mathbb{R}^N$ be an open set. We denote by $H^1(\Omega)$ the first order Sobolev space defined by

$$H^1(\Omega) = \{u \in L^2(\Omega):\ \nabla u \in L^2(\Omega)^N\}$$

where ∇u is defined in the weak sense. The norm on $H^1(\Omega)$ is given by

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega |u|^2\, dx + \int_\Omega |\nabla u|^2\, dx$$

so that $H^1(\Omega)$ is a Hilbert space. Denote by $H^1_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ (the space of all infinitely differentiable functions with compact support in Ω) in $H^1(\Omega)$. If $\Omega = \mathbb{R}^N$, or more generally if $H^1_0(\Omega) = H^1(\mathbb{R}^N)$, this is the case if and only if $\mathbb{R}^N \setminus \Omega$ is a polar set (see Theorem 2.4.4 below) then we will simply denote this space by $H^1(\mathbb{R}^N)$.

Example 1.3.5. a) Consider the form $a_N:\ H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$a_N(u, v) := \int_\Omega \nabla u \nabla v\, dx.$$

Then $(a_N, H^1(\Omega))$ is closed on $L^2(\Omega)$ and the selfadjoint operator associated with $(a_N, H^1(\Omega))$ is the **classical Laplacian with Neumann boundary conditions**.

b) Consider the form $a_D: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$a_D(u,v) := \int_\Omega \nabla u \nabla v \, dx.$$

Then $(a_D, H_0^1(\Omega))$ is closed on $L^2(\Omega)$ and the selfadjoint operator associated with $(a_D, H_0^1(\Omega))$ which we denote by Δ_D is the **Laplacian with Dirichlet boundary conditions**.

Next, let again H be a fixed Hilbert space, $D(a) \hookrightarrow H$ and $a : D(a) \times D(a) \to \mathbb{R}$ be a bilinear positive symmetric form. Let $\widetilde{D}(a)$ be the abstract completion of $D(a)$ with respect to the norm $\|\cdot\|_a$. Let $\tilde{\jmath}: \widetilde{D}(a) \to H$ be the continuous extension of the embedding from $D(a)$ into H.

Definition 1.3.6. *The form $(a, D(a))$ is said to be* **closable** *on H if $\tilde{\jmath}$ is injective.*

Remark 1.3.7. *a) The following criterion is useful to verify the closability of a given form $(a, D(a))$ on H. The form $(a, D(a))$ is closable on H if and only if for each sequence $u_n \in D(a)$ converging to zero in H and such that $\lim_{n,m\to\infty} a(u_n - u_m, u_n - u_m) = 0$, one has $\lim_{n\to\infty} a(u_n, u_n) = 0$.*

b) A necessary and sufficient condition for a symmetric form $(a, D(a))$ to possess a closed extension is that the symmetric form is closable. Then there always exists a smallest closed extension $\tilde{a}$, that is a closed extension whose domain is contained in the domain of all other closed extensions.

Next we give some examples of closable forms.

Example 1.3.8. *a)* Consider the form a_D defined above but now with domain $\mathcal{D}(\Omega)$. Then it is easy to prove that $(a_D, \mathcal{D}(\Omega))$ is closable on $L^2(\Omega)$ and its smallest closed extension is the form $(a_D, H_0^1(\Omega))$.

b) Let $C_c(\bar{\Omega})$ denote the space of continuous functions with compact support in $\bar{\Omega}$. If Ω is bounded then $C_c(\bar{\Omega}) = C(\bar{\Omega})$. Let $\widetilde{H}^1(\Omega)$ be the closure of $H^1(\Omega) \cap C_c(\bar{\Omega})$ in $H^1(\Omega)$. Consider the form a_N with domain $H^1(\Omega) \cap C_c(\bar{\Omega})$. Then it is closable and its smallest closed extension is the form a_N with domain $\widetilde{H}^1(\Omega)$. We call the selfadjoint operator Δ_N associated with this closed form $(a_N, \widetilde{H}^1(\Omega))$, the **Laplacian with Neumann boundary conditions**. If Ω is regular, for example if Ω is bounded and has a continuous boundary (see [75, Theorem 1.4.2.1]), it coincides with the classical Neumann Laplacian. More precisely this is the case if and only if $H^1(\Omega) \cap C_c(\bar{\Omega})$ is dense in $H^1(\Omega)$.

c) Let Ω be a bounded domain with Lipschitz boundary. Let σ be the usual Lebesgue surface measure on $\partial\Omega$. Then the trace application is linear continuous from $H^1(\Omega) \cap C(\bar{\Omega})$ into $L^2(\partial\Omega, \sigma)$ (see Chapter 4). Consider the form a_σ on $L^2(\Omega)$ with domain $H^1(\Omega) \cap C(\bar{\Omega})$ defined by

$$a_\sigma(u,v) := \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} uv \, d\sigma.$$

We show that a_σ is closable. Let $u_n \in H^1(\Omega) \cap C(\bar{\Omega})$ be such that $u_n \to 0$ in $L^2(\Omega)$ and $a_\sigma(u_n - u_m, u_n - u_m)$ converges to 0 as $n, m \to \infty$. Since u_n converges to 0 in $L^2(\Omega)$ and is a Cauchy sequence in $H^1(\Omega)$, it follows that u_n converges to 0 in $H^1(\Omega)$. Now the continuity of the trace application implies that $u_n|_{\partial\Omega}$ converges to 0 in $L^2(\partial\Omega, \sigma)$ and the form a_σ is closable. By the continuity of the trace application again, the completion of $H^1(\Omega) \cap C(\bar{\Omega})$ with respect to the $\|\cdot\|_{a_\sigma}$-norm is the space $\widetilde{H}^1(\Omega)$ which coincides with $H^1(\Omega)$. The selfadjoint operator associated with the closed form $(a_\sigma, H^1(\Omega))$ is called the **Laplacian with Robin boundary condition**.

We shall come back to the forms defined above in Chapters 3 and 4. In particular, we shall prove that it is (always) possible to define the Laplacian with Robin boundary conditions for an arbitrary open set Ω.

We give some examples of forms which are not always closable. For more details, see [55, Theorem 3.1.6].

Example 1.3.9. a) Let μ be a Radon measure on $\mathbb{R}$. We suppose that μ is not absolutely continuous with respect to the Lebesgue measure. Then the following form

$$a(u, v) = \int_{\mathbb{R}} u'(x)v'(x)\, \mu(dx) \quad u, v \in \mathcal{D}(\mathbb{R})$$

is not closable on $L^2(\mathbb{R})$.

b) For a given Borel function $b : \mathbb{R} \to [0, \infty]$ we let

$$R(b) := \left\{ t \in \mathbb{R} : \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{b(y)}\, dy < \infty \text{ for some } \varepsilon > 0 \right\}$$

and

$$S(b) := \mathbb{R} \setminus R(b).$$

We say that b satisfies the **Hamza condition** if

$$b(x) = 0 \text{ a.e. on } S(b). \tag{1.2}$$

Consider again the form defined in a). Then a is closable on $L^2(\mathbb{R})$ if and only if μ is absolutely continuous with respect to the Lebesgue measure and its **density function** satisfies the condition (1.2).

Definition 1.3.10. *Let $(a, D(a))$ and $(b, D(b))$ be two bilinear positive symmetric forms on H. We say that $a \le b$ if and only if $D(b) \subset D(a)$ and $a(u, u) \le b(u, u)$ for all $u \in D(b)$.*

Since each symmetric form is not closable as the last example shows, the following result shows that each positive symmetric form as a closable part. The proof given here is taken from [84, Theorem S.15 p.373]

Theorem 1.3.11 (Reed-Simon). *Let $(a, D(a))$ be a bilinear positive symmetric form on a Hilbert space H. Then there exists a largest closable symmetric form a_r that is smaller than a. We call a_r the* **closable part** *of a.*

Proof. 1) If a is closable, then $a_r = a$.

2) If a is not closable, then we denote by $\widetilde{D}(a)$ the abstract completion of $D(a)$ with respect to the norm $\|\cdot\|_a$. Let $i : \widetilde{D}(a) \to H$ be the continuous extension of the injection $D(a) \hookrightarrow H$. Let P be the orthogonal projection onto $\ker i$ and let $Q = 1 - P$. For $\varphi \in D(a)$, let $j(\varphi)$ be its natural range in $\widetilde{D}(a)$ so that $i \circ j = 1$ and $\|j(\varphi)\|_a^2 = \|\varphi\|_a^2$. For $\varphi, \psi \in D(a)$ define

$$\begin{cases} a_r(\varphi, \psi) = a_1(Qj(\varphi), Qj(\psi)) - (\varphi, \psi)_H \\ a_s(\varphi, \psi) = (Pj(\varphi), Pj(\psi)). \end{cases}$$

We claim that a_r is closable. Indeed,

$$(a_r)_1(\varphi, \psi) := a_r(\varphi, \psi) + (\varphi, \psi)_H = a_1(Qj(\varphi), Qj(\psi))$$

for all $\varphi, \psi \in D(a) = D(a_r)$. The abstract completion of $D(a)$ with respect to the norm $\|\cdot\|_{a_r}$ is $\operatorname{Ran} Q$ (range of Q). Let $\tilde{\imath} : \operatorname{Ran} Q \to H$ be the continuous extension of the injection $D(a) \hookrightarrow H$. One has $\tilde{\imath} = i|_{\operatorname{Ran} Q}$. By construction, $\operatorname{Ran} P \cap \operatorname{Ran} Q = \{0\}$, so that $\ker \tilde{\imath} = \{0\}$ and $\tilde{\imath}$ is injective and we obtain that a_r is closable.

3) Let us prove that $a_r \geq 0$. Since $\operatorname{Ran} P \subset \ker i$, we have $i \circ Q = i(1-P) = i$ and for all $\varphi \in D(a)$ we obtain

$$\|\varphi\|_H^2 = \|i \circ j(\varphi)\|_H^2 = \|iQj(\varphi)\|_H^2 = \|\tilde{\imath}Qj(\varphi)\|_H^2 \leq \|Qj(\varphi)\|_{a_r}^2.$$

Then $a_r(\varphi, \varphi) \geq 0$ for all $\varphi \in D(a)$.

4) Now we prove that a_r is the largest closable part. Let h be closable such that $h \leq a$ and $D(h) = D(a)$. Since h is closable, there exists a unique operator A on $\widetilde{D}(a)$ such that $h_1(\varphi, \psi) = (j(\varphi), Aj(\psi))$. Let $\varphi \in \operatorname{Ran} P \subset \widetilde{D}(a)$. There esists a sequence $\eta_n \in D(a)$ such that $j(\eta_n) := \varphi_n \to \varphi$ as $n \to \infty$. Since i is continuous, we have $i(\varphi_n) = \eta_n \to i(\varphi)$ as $n \to \infty$. Since $\varphi \in \operatorname{Ran} P \subset \ker i$, it follows that $i(\varphi) = 0$. Moreover, since φ_n is a Cauchy sequence in $\widetilde{D}(a)$, η_n is a Cauchy sequence relatively to the norm $\|\cdot\|_h$. Since h is closable, $h(\varphi_n, \varphi_n) \to 0$; i.e., $(\varphi, A\varphi) = 0$. It follows that

$$h(\varphi, \varphi) = h(Q\varphi, Q\varphi) \leq a_r(Q\varphi, Q\varphi)$$

so that $h \leq a_r$. Thus a_r is the largest closable symmetric form smaller than a. $\square$

By the Reed-Simon construction, the closable part of $(a, D(a))$ is obtained by conserving the form domain $D(a)$ and changing a to a smaller form a_r in the sense that $a_r(u, u) \leq a(u, u)$ for all $u \in D(a)$.

Throughout the rest of this chapter, we shall make the topological assumption that

$$\begin{cases} X \ \text{ is a locally compact separable metric space} \\ m \ \text{ is a Radon measure on } \ X \ \text{ such that } \ \operatorname{supp}[m] = X. \end{cases} \tag{1.3}$$

Throughout this chapter and the following chapters for $1 \le p < \infty$ and $u \in L^p(X,m)$, we denote by

$$\|u\|_p := \left(\int_X |u|^p \, dm \right)^{1/p}$$

and $\|u\|_\infty$ for the supremum norm of $u \in L^\infty(X,m)$. We let

$$L^2(X,m)_+ := \{u \in L^2(X,m) : \ u \ge 0 \ m\text{-a.e.}\}$$

and $F_+ = F \cap L^2(X,m)_+$ if F is a subspace of $L^2(X,m)$.

Next, let $(a, D(a))$ be a positive symmetric closed form on $L^2(X,m)$ where $D(a) \hookrightarrow L^2(X,m)$ is assumed to be dense in $L^2(X,m)$. Let $T := (e^{tA})_{t\ge 0}$ be the C_0-semigroup on $L^2(X,m)$ associated with the operator A given by the form a. We introduce the Beurling-Deny criteria which are contained in [39, Theorems 1.3.2 and 1.3.3]. For the non-symmetric case, these criteria have been established by Ouhabaz (see [79, Théorèmes 1.2.2 and 1.2.5]).

Definition 1.3.12. *Let $\mathcal{B} \in \mathcal{L}(L^p(X,m))$ where $1 \le p \le \infty$.*

a) The operator $\mathcal{B}$ is called **positive** *and we write $\mathcal{B} \ge 0$ if $\mathcal{B}\varphi \ge 0$ m-a.e. for all $\varphi \in L^p(X,m)$ with $\varphi \ge 0$ m-a.e.*

b) The operator $\mathcal{B}$ is called L^∞**-contractive** *if*

$$\|\mathcal{B}\varphi\|_\infty \le \|\varphi\|_\infty$$

for all $\varphi \in L^p(X,m) \cap L^\infty(X,m)$.

Theorem 1.3.13 (Beurling-Deny 1). *The following assertions are equivalent.*

(i) $e^{tA} \ge 0$ for every $t \ge 0$.

(ii) $u \in D(a) \Longrightarrow u^+ \in D(a)$ and $a(u^+, u^-) \le 0$.

Theorem 1.3.14 (Beurling-Deny 2). *Assume that $e^{tA} \ge 0$. Then the following assertions are equivalent.*

(i) e^{tA} is L^∞-contractive for every $t \ge 0$.

(ii) $u \in D(a)_+ \Rightarrow u \wedge 1 \in D(a)_+$ and $a(u \wedge 1, u \wedge 1) \le a(u,u)$.

Definition 1.3.15. *a) A C_0-semigroup $T = (T(t))_{t\geq 0}$ on $L^2(X,m)$ is called* **submarkovian** *if $T(t)$ is positive and L^∞-contractive for every $t \geq 0$.*

b) A form $(a, D(a))$ is called a **Dirichlet form** *if $(a, D(a))$ is closed and the associated semigroup is submarkovian.*

Let $(a, D(a))$ be a Dirichlet form on $L^2(X,m)$ and let

$$C_c(X) := \{u \in C(X) : \text{ support of } u \text{ is a compact set}\}.$$

Definition 1.3.16. *The form $(a, D(a))$ is called a* **regular Dirichlet form** *on $L^2(X,m)$ (or on X) if $D(a) \cap C_c(X)$ is dense in $(D(a), \|\cdot\|_a)$ and uniformly dense in $C_c(X)$.*

The following result contained in [39, Theorem 1.4.1] is a direct consequence of the submarkovian property.

Theorem 1.3.17. *If $T = (e^{tA})_{t\geq 0}$ is a symmetric submarkovian semigroup on $L^2(X,m)$ then $L^1(X,m) \cap L^\infty(X,m)$ is invariant under e^{tA}, and e^{tA} may be extended from $L^1(X,m) \cap L^\infty(X,m)$ to a positive contraction semigroup $T_p(t)$ on $L^p(X,m)$ for all $1 \leq p \leq \infty$. These semigroups are strongly continuous if $1 \leq p < \infty$, and are consistent in the sense that*

$$T_p(t)u = T_q(t)u$$

if $u \in L^p(X,m) \cap L^q(X,m)$. They are selfadjoint in the sense that

$$T_p(t)^* = T_q(t)$$

if $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Part I

General Robin Boundary Conditions on Arbitrary Domains

Chapter 2

Classical and Relative Capacities

In this chapter, we will define two notions of capacities. The first one which we call *classical capacity* is well-known and has been considered by many authors. The second one which we call *relative capacity* is new and is the correct one for studying the fine regularity of functions in some subspace of the first Sobolev space $H^1(\Omega)$ for some open set Ω in $\mathbb{R}^N$. Throughout this chapter the underlying field is $\mathbb{R}$.

2.1 Classical Capacity.

Consider the closed form $(a, H^1(\mathbb{R}^N))$ defined by

$$a(u,v) = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx.$$

Since $\mathcal{D}(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$ and uniformly dense in $C_c(\mathbb{R}^N)$, the symmetric form $(a, H^1(\mathbb{R}^N))$ is a regular Dirichlet form on $L^2(\mathbb{R}^N)$. Thus we can define a notion of capacity with some regularity properties.

Definition 2.1.1. *a) The* **classical capacity** *which we denote by* Cap *is defined on subsets of* $\mathbb{R}^N$ *by: for* $A \subset \mathbb{R}^N$ *open we set:*

$$\operatorname{Cap}(A) := \inf\{\|u\|^2_{H^1(\mathbb{R}^N)} : \ u \in H^1(\mathbb{R}^N) : \ u \geq 1 \text{ a.e. on } A\}.$$

For arbitrary $A \subset \mathbb{R}^N$ *we set:*

$$\operatorname{Cap}(A) := \inf\{\operatorname{Cap}(B) : \ B \text{ open } : A \subset B \subset \mathbb{R}^N\}.$$

b) A set $N \subset \mathbb{R}^N$ *is called* **polar** *if* $\operatorname{Cap}(N) = 0$.

The classical capacity is an outer measure; i.e. it has the following properties.

- $\mathrm{Cap}(\emptyset) = 0$.
- If A_n is an arbitrary sequence of subsets of $\mathbb{R}^N$, then

$$\mathrm{Cap}(\bigcup_{n\geq 1} A_n) \leq \sum_{n\geq 1} \mathrm{Cap}(A_n).$$

However, Cap is not a Borel measure.

Theorem 2.1.2. *The classical capacity is a* **Choquet capacity***; i.e. it has the following properties.*

a) $A \subset B \Rightarrow \mathrm{Cap}(A) \leq \mathrm{Cap}(B)$.

b) If (K_n) *is a decreasing sequence of compact subsets of* $\mathbb{R}^N$*, then*

$$\mathrm{Cap}(\bigcap_{n\geq 1} K_n) = \inf_n \mathrm{Cap}(K_n).$$

c) If (A_n) *is an increasing sequence of arbitrary subsets of* $\mathbb{R}^N$*, then*

$$\mathrm{Cap}(\bigcup_{n\geq 1} A_n) = \sup_n \mathrm{Cap}(A_n).$$

It holds that for every Borel set $A \subset \mathbb{R}^N$

$$\mathrm{Cap}(A) = \sup\{\mathrm{Cap}(K) : K \text{ compact }, K \subset A\}. \tag{2.1}$$

Let A be a subset of $\mathbb{R}^N$. A statement depending on $x \in A$ is said to hold **quasi-everywhere** (**q.e.**) on A if there exists a polar set $N \subset A$ such that the statement is true for every $x \in A \setminus N$.

We call a function u **quasi-continuous** (**q.c.**) if for every $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^N$ such that $\mathrm{Cap}(G) < \varepsilon$ and $u|_{\mathbb{R}^N \setminus G}$ is continuous.

Theorem 2.1.3. *a) Every* $u \in H^1(\mathbb{R}^N)$ *admits a* **quasi-continuous version** $\tilde{u}$ *such that* $\tilde{u} = u$ *a.e. on* $\mathbb{R}^N$.

b) If $\lim_{n\to\infty} u_n = u$ *in* $H^1(\mathbb{R}^N)$*, then there exist a subsequence* (u_{n_k}) *such that* $\lim_{k\to\infty} \tilde{u}_{n_k}(x) = \tilde{u}(x)$ *q.e.*

c) Let $O \subset \mathbb{R}^N$ *be an open set and* u *be quasi-continuous. If* $u \geq 0$ *a.e. on* O*, then* $u \geq 0$ *q.e. on* O.

Remark 2.1.4. *For* $u \in H^1(\mathbb{R}^N)$ *the quasi-continuous version* $\tilde{u}$ *of* u *is unique q.e. Moreover* $\tilde{u}$ *can be chosen* **Borel measurable** *(see the proof of [23, Proposition 8.2.1]).*

By definition of the classical capacity, it is clear that $|A| \leq \mathrm{Cap}(A)$ for every $A \subset \mathbb{R}^N$, where $|\cdot|$ denotes the Lebesgue measure. Therefore every polar set has zero Lebesgue measure. The following result says that every polar set has zero s-dimensional Hausdorff measure for all $s > N-2$. In particular every polar set has zero $(N-1)$-dimensional Hausdorff measure.

Theorem 2.1.5. *Let $A \in \mathcal{B}(\mathbb{R}^N)$. If $\mathrm{Cap}(A) = 0$, then $\mathcal{H}^s(A) = 0$ for all $s > N-2$.*

Proof. 1) Assume that $\mathrm{Cap}(A) = 0$. Then for all $n \geq 1$ there exists $u_n \in H^1(\mathbb{R}^N)$ such that $A \subset \{u_n \geq 1\}^o$ and $\|u_n\|^2_{H^1(\mathbb{R}^N)} \leq \frac{1}{2^n}$. By Theorem 2.1.3 a), we may assume that the u_n are quasi-continuous. Let $v := \sum_{n=1}^\infty u_n$. Then

$$\|v\|_{H^1(\mathbb{R}^N)} \leq \sum_{n=1}^\infty \|u_n\|_{H^1(\mathbb{R}^N)} < \infty.$$

Thus $v \in H^1(\mathbb{R}^N)$ and is quasi-continuous.

2) Note $A \subset \{v \geq m\}^o$ for all $m \geq 1$. Fix any $a \in A$. Then for r small enough such that $B(a,r) \subset \{v \geq m\}^o$, $(v)_{a,r} \geq m$, therefore $(v)_{a,r} \to \infty$ as $r \to 0$ where

$$(v)_{a,r} := \frac{1}{|B(a,r)|} \int_{B(a,r)} v(x)\, dx.$$

We claim that for each $a \in A$,

$$\limsup_{r\to 0} \frac{1}{r^s} \int_{B(a,r)} |\nabla v|^2\, dx = \infty.$$

In fact, let $a \in A$ and suppose

$$\limsup_{r\to 0} \frac{1}{r^s} \int_{B(a,r)} |\nabla v|^2\, dx < \infty.$$

Then there exists a constant $M < \infty$ such that

$$\frac{1}{r^s} \int_{B(a,r)} |\nabla v|^2\, dx \leq M$$

for all $0 < r \leq 1$. For $0 < r \leq 1$, the Poincaré inequality on balls (see [43, Theorem 2, p.141]) gives

$$\frac{1}{|B(a,r)|} \int_{B(a,r)} |v - (v)_{a,r}|^2\, dx \leq cr^2 \frac{1}{|B(a,r)|} \int_{B(a,r)} |\nabla v|^2\, dx \leq cr^\theta$$

where $\theta = s - (N-2)$. Thus

$$\begin{aligned} |(v)_{a,r/2} - (v)_{a,r}| &= \frac{1}{|B(a,r/2)|}\left|\int_{B(a,r/2)} (v-(v)_{a,r})\, dx\right| \\ &\le 2^N \frac{1}{|B(a,r)|}\int_{B(a,r)} |v-(v)_{a,r}|\, dx \\ &\le 2^N \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} |v-(v)_{a,r}|^2\, dx\right)^{1/2} \\ &\le cr^{\frac{\theta}{2}}. \end{aligned}$$

Hence if $k > j$,

$$\begin{aligned} \left|(v)_{a,1/2^k} - (v)_{a,1/2^j}\right| &\le \sum_{l=j+1}^{k} \left|(v)_{a,1/2^l} - (v)_{a,1/2^{l-1}}\right| \\ &\le c\sum_{l=j+1}^{k}\left(\frac{1}{2^{l-1}}\right)^{\frac{\theta}{2}}. \end{aligned}$$

This last sum is the tail of a geometric series and so $\left\{(v)_{a,1/2^k}\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Thus $(v)_{a,1/2^k} \not\to \infty$, a contradiction and the claim is proved.

3) Consequently,

$$\begin{aligned} A &\subset \{a \in \mathbb{R}^N : \limsup_{r\to 0}\frac{1}{r^s}\int_{B(a,r)} |\nabla v|^2\, dx = +\infty\} \\ &\subset \{a \in \mathbb{R}^N : \limsup_{r\to 0}\frac{1}{r^s}\int_{B(a,r)} |\nabla v|^2\, dx > 0\}. \end{aligned}$$

But since $|\nabla u|^2$ is integrable, by [43, Theorem 3, p.77], $\mathcal{H}^s(A) = 0$. □

2.2 Relative Capacity.

Throughout this section, Ω will denote an open set in $\mathbb{R}^N$ and $\widetilde{H}^1(\Omega)$ the closure of $H^1(\Omega) \cap C_c(\bar{\Omega})$ in $H^1(\Omega)$. To define a *Choquet capacity* with some regularity properties, we need a regular Dirichlet form. If Ω is a set such that the Lebesgue measure of its boundary $\mathcal{L}^N(\partial\Omega) := |\partial\Omega| > 0$, then we shall consider the measure m with support $\bar{\Omega}$ defined by: for $A \in \mathcal{B}(\bar{\Omega})$ we let $m(A) := |A \cap \Omega|$. With this consideration, $L^2(\Omega) = L^2(\bar{\Omega}, m)$. Moreover, by Stone-Weierstrass' Theorem the space $H^1(\Omega) \cap C_c(\bar{\Omega})$ is uniformly dense in $C_c(\bar{\Omega})$. Therefore the form $(a_N, \widetilde{H}^1(\Omega))$ is a regular Dirichlet form on $L^2(\bar{\Omega}, m)$ (or on $\bar{\Omega}$). Thus, throughout the following, a.e. will mean m-a.e.

Definition 2.2.1. *a) The* **relative capacity** *which we denote by* $\mathrm{Cap}_{\bar{\Omega}}$ *is defined on subsets of* $\bar{\Omega}$ *by: for* $A \subset \bar{\Omega}$ *relatively open (i.e. open with respect to the topology of* $\bar{\Omega}$*) we set:*

$$\mathrm{Cap}_{\bar{\Omega}}(A) := \inf\{\|u\|^2_{H^1(\Omega)} :\ u \in \widetilde{H}^1(\Omega) :\ u \geq 1 \text{ a.e. on } A\}.$$

For arbitrary $A \subset \bar{\Omega}$ *we set:*

$$\mathrm{Cap}_{\bar{\Omega}}(A) := \inf\{\mathrm{Cap}_{\bar{\Omega}}(B) :\ B \text{ relatively open } : A \subset B \subset \bar{\Omega}\}.$$

b) A set $N \subset \bar{\Omega}$ *is called* **relatively polar** *if* $\mathrm{Cap}_{\bar{\Omega}}(N) = 0$.

The relative capacity is also an outer measure (but not a Borel measure) and a Choquet capacity. Then the properties in Theorem 2.1.2 are satisfied for $\mathrm{Cap}_{\bar{\Omega}}$ in place of Cap and $\bar{\Omega}$ in place of $\mathbb{R}^N$.

Similarly to the classical capacity, a statement depending on $x \in A \subset \bar{\Omega}$ is said to hold **relatively quasi-everywhere** (**r.q.e.**) on A if there exists a relatively polar set $N \subset A$ such that the statement is true for every $x \in A \setminus N$.

Now we may consider functions in $\widetilde{H}^1(\Omega)$ as defined on $\bar{\Omega}$. We call a function $u :\ \bar{\Omega} \to \mathbb{R}$ **relatively quasi-continuous** (**r.q.c.**) if for every $\varepsilon > 0$ there exists a relatively open set $G \subset \bar{\Omega}$ such that $\mathrm{Cap}_{\bar{\Omega}}(G) < \varepsilon$ and $u|_{\bar{\Omega}\setminus G}$ is continuous.

In Theorem 2.1.3, replacing Cap by $\mathrm{Cap}_{\bar{\Omega}}$ and $\mathbb{R}^N$ by $\bar{\Omega}$, all the properties are satisfied.

For $B \subset \bar{\Omega}$, we let

$$\mathcal{L}_B := \{u \in \widetilde{H}^1(\Omega) :\ \tilde{u} \geq 1 \ \text{ r.q.e. on } \ B\}$$

where $\tilde{u}$ denote the relatively quasi-continuous version of u. By Theorem 2.1.3 c) applied to the relative capacity, if B is relatively open, then

$$\mathcal{L}_B = \{u \in \widetilde{H}^1(\Omega) :\ \tilde{u} \geq 1 \ \text{ a.e. on } \ B\}.$$

The following properties of the relative capacity are properties of a Choquet capacity. The proof we give here is an adaptation of the proof contained in [55, Theorem 2.1.5] for a general abstract Choquet capacity.

Lemma 2.2.2. *Fix an arbitrary set* $B \subset \bar{\Omega}$.

a) If $\mathcal{L}_B \neq \emptyset$*, then there exists a unique element* $e_B \in \mathcal{L}_B$ *minimizing the norm of* $\widetilde{H}^1(\Omega)$ *and* e_B *satisfies*

$$\mathrm{Cap}_{\bar{\Omega}}(B) = \|e_B\|^2_{H^1(\Omega)}. \tag{2.2}$$

b) e_B *is a unique element of* $\widetilde{H}^1(\Omega)$ *satisfying*

$$0 \leq e_B \leq 1 \ \text{ a.e. and } \ \tilde{e_B} = 1 \ \text{ r.q.e. on } \ B \tag{2.3}$$

and

$$a_1(e_B, v) \geq 0 \ \ \forall\, v \in \widetilde{H}^1(\Omega),\ \tilde{v} \geq 0 \ \text{ r.q.e. on } \ B. \tag{2.4}$$

Proof. a) **First case**. Assume that B is relatively open. Then $\mathcal{L}_B$ is a closed convex subset of $\widetilde{H}^1(\Omega)$. Since for all $u, v \in \widetilde{H}^1(\Omega)$,

$$\left\| \frac{u-v}{2} \right\|^2_{H^1(\Omega)} + \left\| \frac{u+v}{2} \right\|^2_{H^1(\Omega)} = \frac{1}{2} \|u\|^2_{H^1(\Omega)} + \frac{1}{2} \|v\|^2_{H^1(\Omega)}, \tag{2.5}$$

we have that any minimizing sequence ($\lim_{n\to\infty} \|u_n\|^2_{H^1(\Omega)} = \mathrm{Cap}_{\bar{\Omega}}(B)$) is convergent to an element $e_B \in \mathcal{L}_B$ satisfying $\|e_B\|^2_{H^1(\Omega)} = \mathrm{Cap}_{\bar{\Omega}}(B)$ and that such e_B is unique.

Second case. Assume that B is arbitrary. If $\mathcal{L}_B$ is nonempty, it is a closed convex subset of $\widetilde{H}^1(\Omega)$ on account of properties in Theorem 2.1.3 for the relative capacity. As in the first case, we find a unique element $e_B \in \mathcal{L}_B$ such that

$$\|e_B\|^2_{H^1(\Omega)} \le \|u\|^2_{H^1(\Omega)} \quad \forall\, u \in \mathcal{L}_B.$$

For every $\varepsilon > 0$, there exists $A \subset \bar{\Omega}$ relatively open such that $B \subset A$ and $\mathrm{Cap}_{\bar{\Omega}}(A) < \mathrm{Cap}_{\bar{\Omega}}(B) + \varepsilon$. Since $e_A \in \mathcal{L}_B$, we have that

$$\mathrm{Cap}_{\bar{\Omega}}(A) = \|e_A\|^2_{H^1(\Omega)} \ge \|e_B\|^2_{H^1(\Omega)}.$$

Thus

$$\|e_B\|^2_{H^1(\Omega)} \le \mathrm{Cap}_{\bar{\Omega}}(B).$$

To prove the converse inequality, fix a r.q.c. version $\tilde{e_B}$ of e_B. For every $\varepsilon > 0$, choose a relatively open set A_ε such that $\mathrm{Cap}_{\bar{\Omega}}(A_\varepsilon) < \varepsilon$, $\tilde{e_B}|_{\bar{\Omega}\setminus A_\varepsilon}$ is continuous and $\tilde{e_B} \ge 1$ on $B \cap (\bar{\Omega} \setminus A_\varepsilon)$. Now the set

$$G_\varepsilon := \{x \in \bar{\Omega} \setminus A_\varepsilon : \ \tilde{e_B} > 1 - \varepsilon\} \bigcup A_\varepsilon$$

is relatively open and $B \subset G_\varepsilon$. Moreover, $e_B + e_{A_\varepsilon} \ge 1 - \varepsilon$ a.e. on G_ε. Therefore

$$\begin{aligned} \mathrm{Cap}_{\bar{\Omega}}(B) &\le \mathrm{Cap}_{\bar{\Omega}}(G_\varepsilon) \le (1-\varepsilon)^2 \|e_B + e_{A_\varepsilon}\|^2_{H^1(\Omega)} \\ &\le (1-\varepsilon)^2 \left(\|e_B\|_{H^1(\Omega)} + \|e_{A_\varepsilon}\|_{H^1(\Omega)} \right)^2 \\ &\le (1-\varepsilon)^2 \left(\|e_B\|_{H^1(\Omega)} + \sqrt{\varepsilon} \right)^2. \end{aligned}$$

By letting $\varepsilon \downarrow 0$, we obtain that

$$\mathrm{Cap}_{\bar{\Omega}}(B) \le \|e_B\|^2_{H^1(\Omega)}$$

and the proof of a) is complete.

b) By the submarkovian property of the form $(a_N, \widetilde{H}^1(\Omega))$, $u := (0 \vee e_B) \wedge 1 \in \mathcal{L}_B$ and

$$\|u\|^2_{H^1(\Omega)} \le \|e_B\|^2_{H^1(\Omega)} = \mathrm{Cap}_{\bar{\Omega}}(B).$$

Thus $u = e_B$ which proves (2.3). To prove (2.4), if v has the stated property, then $e_B + \varepsilon v \in \mathcal{L}_B$ and $\|e_B + \varepsilon v\|^2_{H^1(\Omega)} \geq \|e_B\|^2_{H^1(\Omega)}$ for every $\varepsilon > 0$. Then

$$\|e_B + \varepsilon v\|^2_{H^1(\Omega)} = \|e_B\|^2_{H^1(\Omega)} + \varepsilon^2 \|v\|^2_{H^1(\Omega)} + 2\varepsilon a_1(e_B, v) \geq \|e_B\|^2_{H^1(\Omega)}$$

and we obtain that

$$a_1(e_B, v) \geq 0.$$

Conversely, suppose that $u \in \widetilde{H}^1(\Omega)$ satisfies (2.4). Then $u \in \mathcal{L}_B$ and $\tilde{w} - \tilde{u} \geq 0$ r.q.e. on B for every $w \in \mathcal{L}_B$ since u is minimal. Hence

$$\|w\|^2_{H^1(\Omega)} = \|u + (w-u)\|^2_{H^1(\Omega)} \geq \|u\|^2_{H^1(\Omega)}$$

for every $w \in \mathcal{L}_B$ proving that $u = e_B$. □

It follows from the preceding lemma that for every $B \subset \bar{\Omega}$,

$$\mathrm{Cap}_{\bar{\Omega}}(B) = \inf\{\|u\|^2_{H^1(\Omega)} : \; u \in \mathcal{L}_B\}.$$

The next two results give an equivalent definition of $\mathrm{Cap}_{\bar{\Omega}}$ for compact subsets of $\bar{\Omega}$.

Proposition 2.2.3. *Let $K \subset \bar{\Omega}$ be a compact set. Then*

$$\mathrm{Cap}_{\bar{\Omega}}(K) = \inf\{\|u\|^2_{H^1(\Omega)} : \; u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \; u(x) \geq 1 \; \forall \, x \in K\}.$$

Proof. Let

$$M := \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \; u(x) \geq 1 \; \forall \, x \in K\}.$$

Consider a sequence $u_n \in M$ minimizing the norm of $H^1(\Omega)$. By virtue of (2.5), u_n converges to some $u_0 \in \widetilde{H}^1(\Omega)$. In view of Lemma 2.2.2, it is sufficient to prove that $u_0 = e_K$ by checking the conditions (2.3) and (2.4) for u_0. By [55, Lemma 2.2.6], for (2.4), it is sufficient to verify the inequality for all functions $v \in H^1(\Omega) \cap C_c(\bar{\Omega})$ satisfying $v \geq 0$ on K. Let v satisfy this condition. For every $\varepsilon > 0$,

$$\|u_n + \varepsilon v\|^2_{H^1(\Omega)} \geq \|u_0\|^2_{H^1(\Omega)},$$

and we see that u_0 satisfies

$$a_1(u_0, v) \geq 0 \;\; \forall \, v \in H^1(\Omega) \cap C_c(\bar{\Omega}), \; v \geq 0 \;\text{ on }\; K,$$

by letting $n \to \infty$ and $\varepsilon \downarrow 0$. Thus u_0 satisfies (2.4). Noting that $v_n := (0 \vee u_n) \wedge 1 \in M$ is also a minimizing sequence, we obtain (2.3) and the proof is complete. □

Theorem 2.2.4. *Let $K \subset \bar{\Omega}$ be a compact set and*

$$N := \left\{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : u = 1 \text{ on } K, 0 \leq u \leq 1\right\}.$$

Then

$$\mathrm{Cap}_{\bar{\Omega}}(K) = \inf\{\|u\|^2_{H^1(\Omega)} : \; u \in N\}. \tag{2.6}$$

Proof. Let

$$M := \left\{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : u \geq 1 \text{ on } K\right\}.$$

By Proposition 2.2.3,

$$\mathrm{Cap}_{\bar{\Omega}}(K) = \inf\{\|u\|^2_{H^1(\Omega)} : u \in M\}. \tag{2.7}$$

Set

$$Ca(K) := \inf\{\|u\|^2_{H^1(\Omega)} : u \in N\}.$$

It suffices to prove that $Ca(K) = \mathrm{Cap}_{\bar{\Omega}}(K)$ where $\mathrm{Cap}_{\bar{\Omega}}(K)$ is given by (2.7). Since $N \subset M$, it suffices to prove that $Ca(K) \leq \mathrm{Cap}_{\bar{\Omega}}(K)$. Let $\varepsilon \in (0,1)$ and $u \in M$ be such that

$$\|u\|^2_{H^1(\Omega)} \leq \mathrm{Cap}_{\bar{\Omega}}(K) + \varepsilon.$$

Let (λ_m) be a sequence of functions in $C^\infty(\mathbb{R})$ such that

$$\begin{cases} 0 \leq \lambda_m'(t) \leq 1 + \frac{1}{m}, \\ \lambda_m(t) = 0 \text{ if } t \leq 0, \\ \lambda_m(t) = 1 \text{ if } t \geq 1, \\ 0 \leq \lambda_m(t) \leq 1 \quad \forall\, t. \end{cases} \tag{2.8}$$

Since $\lambda_m \circ u \in N$, it follows that

$$\begin{aligned} Ca(K) &\leq \int_\Omega \left[(\lambda_m'(u(x)))^2 |\nabla u|^2 + |\lambda_m(u(x))|^2\right]\, dx \\ &\leq \left(1 + \frac{1}{m}\right) \int_\Omega \left[|\nabla u|^2 + |u|^2\right]\, dx. \end{aligned}$$

Passing to the limit as $m \to \infty$, we obtain

$$Ca(K) \leq \|u\|^2_{H^1(\Omega)} \leq \mathrm{Cap}_{\bar{\Omega}}(K) + \varepsilon.$$

Passing to the limit as $\varepsilon \downarrow 0$ we conclude that $Ca(K) \leq \mathrm{Cap}_{\bar{\Omega}}(K)$. $\square$

The following two results give some inequalities which are a consequence of the definition of the relative capacity.

Proposition 2.2.5. *Assume that Ω is bounded. Let $u \in H^1(\Omega) \cap C(\bar{\Omega})$ and*

$$E_t := \left\{x \in \bar{\Omega} : |u(x)| \geq t\right\}.$$

Then there exists a constant $c > 0$ such that

$$\int_0^\infty \mathrm{Cap}_{\bar{\Omega}}(E_t)\, d(t^2) \leq c\|u\|^2_{H^1(\Omega)}. \tag{2.9}$$

Proof. Let $u \in H^1(\Omega) \cap C_c(\bar{\Omega})$ and $j \in \mathbb{Z}$. Since $\mathrm{Cap}_{\bar{\Omega}}(E_t)$ is a decreasing function of t, it suffices to prove that

$$S := \sum_{j=-\infty}^{+\infty} 2^{2j} \,\mathrm{Cap}_{\bar{\Omega}}(E_{2^j}) \leq c\|u\|^2_{H^1(\Omega)}.$$

Note that for $j \in \mathbb{Z}$, E_{2^j} is a compact subset of $\bar{\Omega}$ and $E_{2^j} = \emptyset$ for j large enough. Let $\varepsilon \in (0,1)$ and λ_ε be as in (2.8). Set $u_j(x) := \lambda_\varepsilon \left(2^{1-j}|u(x)| - 1\right)$. Then $u_j \in H^1(\Omega) \cap C(\bar{\Omega})$. Moreover, $u_j = 1$ on $E_{2^j}, 0 \leq u \leq 1$ and $\mathrm{supp}[u_j] \subset E_{2^{j-1}}$. Thus

$$S \leq \sum_{j=-\infty}^{+\infty} 2^{2j} \int_{E_{2^{j-1}}} \left[|\nabla u_j|^2 + |u_j|^2\right] \,dx.$$

We obtain

$$\begin{aligned}
\sum_{j=-\infty}^{+\infty} 2^{2j} \int_{E_{2^{j-1}}} \left[|\nabla u_j|^2 + |u_j|^2\right] \,dx &= \sum_{j=-\infty}^{+\infty} 2^{2j} \int_{E_{2^{j-1}} \setminus E_{2^j}} |\nabla u_j|^2 \,dx + \\
&+ \sum_{j=-\infty}^{+\infty} 2^{2j} \int_{E_{2^j}} 1 \,dx.
\end{aligned}$$

Since $\nabla u_j = \lambda'_\varepsilon \left(2^{1-j}|u(x)| - 1\right) 2^{1-j} \nabla u(x) \,\mathrm{sgn}\, u$, we have

$$\begin{aligned}
\sum_{j=-\infty}^{+\infty} 2^{2j} \int_{E_{2^{j-1}} \setminus E_{2^j}} |\nabla u_j|^2 \,dx &= \sum_{j=-\infty}^{+\infty} 2^{2j} 2^{2-2j}. \\
&\cdot \int_{E_{2^{j-1}} \setminus E_{2^j}} \left(\lambda_\varepsilon(2^{1-j}|u(x)| - 1)\right)^2 |\nabla u|^2 \,dx \\
&\leq 2(1+\varepsilon)^2 \sum_{j=-\infty}^{+\infty} \int_{E_{2^{j-1}} \setminus E_{2^j}} |\nabla u|^2 \,dx.
\end{aligned}$$

Setting $A_j := E_{2^{j-1}} \setminus E_{2^j}$, we have $A_j \cap A_i = \emptyset$ for $i \neq j$ and $\bigcup_{j=-\infty}^{+\infty} A_j = \bar{\Omega}$. Then

$$\sum_{j=-\infty}^{+\infty} 2^{2j} \int_{E_{2^{j-1}} \setminus E_{2^j}} |\nabla u_j|^2 \,dx \leq 2(1+\varepsilon)^2 \int_\Omega |\nabla u(x)|^2 \,dx.$$

Moreover,

$$\begin{aligned}
\sum_{j=-\infty}^{+\infty} 2^{2j} \int_{E_{2^j}} 1 \,dx = \sum_{j=-\infty}^{+\infty} 2^{2j} |E_{2^j}| &\leq c \int_0^\infty |E_t| \,d(t^2) \\
&\leq c \int_\Omega |u(x)|^2 \,dx.
\end{aligned}$$

Thus

$$\int_0^\infty \mathrm{Cap}_{\bar{\Omega}}(E_t)\, d(t^2) \le 2(1+\varepsilon)^2 \|\nabla u\|_2^2 + c\|u\|_2^2.$$

Letting $\varepsilon \to 0$ we obtain (2.9). $\square$

Proposition 2.2.6. *Assume that Ω is bounded. Let μ be a finite Borel measure on $\partial\Omega$ and $p \ge 2$. Then the following assertions are equivalent.*

(i) There exists a constant $c > 0$ such that

$$\mu(K)^{2/p} \le c\, \mathrm{Cap}_{\bar{\Omega}}(K) \tag{2.10}$$

for every compact set $K \subset \partial\Omega$.

(ii) There exists a constant $c_1 > 0$ such that

$$\|u\|_{L^p(\partial\Omega,\mu)} \le c_1 \|u\|_{H^1(\Omega)} \tag{2.11}$$

for all $u \in H^1(\Omega) \cap C(\bar{\Omega})$.

The proof uses the following lemma whose is taken from [33, Lemma 7.2.6].

Lemma 2.2.7. *Suppose that $p \ge 1$ and g is a nonnegative nonincreasing function on $(0,\infty)$. Then*

$$\left(\int_0^\infty [g(s)]^p\, d(s^p)\right)^{1/p} \le \int_0^\infty g(s)\, ds. \tag{2.12}$$

Proof. To prove (2.12), observe that

$$sg(s) = \int_0^s g(s)\, dt \le \int_0^s g(t)\, dt$$

where we use that g is nonnegative and nonincreasing. Moreover, the function defined by $s \mapsto \int_0^s g(t)\, dt$ is absolutely continuous and hence

$$p\left(\int_0^s g(t)\, dt\right)^{p-1} g(s) = \frac{d}{ds}\left(\int_0^s g(t)\, dt\right)^p$$

for almost all $s \ge 0$. Therefore,

$$\begin{aligned}
\int_0^\infty [g(s)]^p\, d(s^p) &= p\int_0^\infty [sg(s)]^{p-1} g(s)\, ds \\
&\le p\int_0^\infty \left(\int_0^s g(t)\, dt\right)^{p-1} g(s)\, ds \\
&= \left(\int_0^\infty g(t)\, dt\right)^p
\end{aligned}$$

proving (2.12). $\square$

Proof of Proposition 2.2.6. (i)$\Rightarrow$ (ii). Let $p \geq 2$, $u \in H^1(\Omega) \cap C(\bar{\Omega})$ and $E_t := \{x \in \partial\Omega : \ |u(x)| \geq t\}$. Then E_t is a compact subset of $\partial\Omega$ and by (2.10), $\mu(E_t)^{2/p} \leq c\,\mathrm{Cap}_{\bar{\Omega}}(E_t)$. Using Lemma 2.2.7 and Proposition 2.2.5, we obtain the following estimates:

$$
\begin{aligned}
\left(\int_{\partial\Omega} |u(x)|^p \, d\mu\right)^{2/p} &= \left(\int_0^\infty \mu(E_t)\, d(t^p)\right)^{2/p} \\
&\leq \int_0^\infty \mu(E_t)^{2/p}\, d(t^2) \\
&\leq c\int_0^\infty \mathrm{Cap}_{\bar{\Omega}}(E_t)\, d(t^2) \\
&\leq c_1^2 \|u\|^2_{H^1(\Omega)}.
\end{aligned}
$$

(ii) $\Rightarrow$ (i). Assume that (2.11) holds and let $K \subset \partial\Omega$ be a compact set. Let $u \in H^1(\Omega) \cap C(\bar{\Omega})$ be such that $u \geq 1$ on K. Minimizing the inequality (2.11) over such u, we obtain (2.10). $\square$

2.3 Relation Between the two Notions of Capacity.

By definition of the two capacities, we have

$$
\mathrm{Cap}_{\bar{\Omega}}(A) \leq \mathrm{Cap}(A) \tag{2.13}
$$

for every $A \subset \bar{\Omega}$. Indeed, let $A \subset \bar{\Omega}$ and set

$$
\begin{aligned}
M &:= \{u \in H^1(\mathbb{R}^N) : \ u \geq 1 \ \text{ a.e. in a neighborhood of } \ A\} \\
N &:= \{u \in \widetilde{H}^1(\Omega) : \ u \geq 1 \ \text{ a.e. in a relatively neighborhood of } \ A\}.
\end{aligned}
$$

We claim that $M \subset N$. In fact, let $U \in M$. Since $U \in H^1(\mathbb{R}^N)$, there exists a sequence $U_n \in \mathcal{D}(\mathbb{R}^N)$ such that U_n converges to U in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Let $u_n := U_n|_\Omega$ and $u := U|_\Omega$. Then $u_n \in C_c^\infty(\bar{\Omega}) \subset H^1(\Omega) \cap C_c(\bar{\Omega})$ and

$$
\|u_n - u\|_{H^1(\Omega)} \leq \|U_n - U\|_{H^1(\mathbb{R}^N)}
$$

which converges to 0 as $n \to \infty$ and thus $u \in \widetilde{H}^1(\Omega)$. Since $U \geq 1$ a.e. in a neighborhood of A, there exists an open set $O \subset \mathbb{R}^N$ such that $A \subset O$ and $U \geq 1$ a.e. on O. Finally we obtain that $u \geq 1$ a.e. on $O \cap \bar{\Omega}$ and $A \subset O \cap \bar{\Omega}$ which is a relatively neighborhood of A and thus $u \in N$.

Since $M \subset N$, it follows that

$$
\inf\{\|u\|^2_{H^1(\Omega)} : \ u \in N\} \leq \inf\{\|u\|^2_{H^1(\Omega)} : \ u \in M\}. \tag{2.14}
$$

Since for every $u \in M$ we have $\|u\|_{H^1(\Omega)} \leq \|u\|_{H^1(\mathbb{R}^N)}$, it follows that

$$\inf\{\|u\|^2_{H^1(\Omega)} : \ u \in M\} \leq \inf\{\|u\|^2_{H^1(\mathbb{R}^N)} : \ u \in M\}. \tag{2.15}$$

Now (2.14) and (2.15) give (2.13). As a consequence, if $A \subset \bar{\Omega}$ and $\mathrm{Cap}(A) = 0$ then $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$. We prove in this section that the converse is always true for subsets of Ω, but if Ω is irregular there may exist relatively polar subsets of $\partial\Omega$ which are not polar.

Proposition 2.3.1. *Let $A \subset \Omega$. Then $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ if and only if $\mathrm{Cap}(A) = 0$.*

Proof. We show that if $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ then $\mathrm{Cap}(A) = 0$ for every $A \subset \Omega$.

First case. Assume that there exists an open bounded set ω such that $A \subset \bar{\omega} \subset \Omega$. Since $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ there exist open sets $O_k \subset \mathbb{R}^n$, $u_k(x) \geq 1$ on $O_k \cap \Omega$ and $\|u_k\|^2_{H^1(\Omega)} \leq \frac{1}{k}$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be such that $\mathrm{supp}[\varphi] \subset \Omega$ and $\varphi = 1$ on ω. Let $v_k = \varphi u_k$ on Ω and $v_k = 0$ on $\mathbb{R}^n \setminus \Omega$. Then $v_k \in H^1(\mathbb{R}^n)$, $v_k = 1$ on $O_k \cap \omega$ and $\|v_k\|_{H^1(\mathbb{R}^n)} \to 0$ as $k \to \infty$. Thus $\mathrm{Cap}(A) = 0$.

Second case. Assume that $A \subset \Omega$ is arbitrary. Take open bounded sets ω_k such that $\bar{\omega}_k \subset \omega_{k+1} \subset \Omega$ and $\bigcup_{k\in\mathbb{N}} \omega_k = \Omega$. It follows from the first case that $\mathrm{Cap}(A \cap \omega_k) = 0$. Hence $\mathrm{Cap}(A) = \lim_{k\to\infty} \mathrm{Cap}(A \cap \omega_k) = 0$ by the property c) of Theorem 2.1.2 for the relative capacity. □

Definition 2.3.2. *a) We say that $\widetilde{H}^1(\Omega)$ has the* **extension property** *if for each $u \in \widetilde{H}^1(\Omega)$ there exists $U \in H^1(\mathbb{R}^N)$ such that $U|_\Omega = u$.*

b) We say that $\widetilde{H}^1(\Omega)$ has the **continuous extension property** *if for each $u \in \widetilde{H}^1(\Omega) \cap C_c(\bar{\Omega})$ there exists $U \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ such that $U|_\Omega = u$ and $\|U\|_{H^1(\mathbb{R}^N)} \leq c\|u\|_{H^1(\Omega)}$ for some constant $c > 0$ independent of u.*

Notice that in the preceding definition the existence of $U \in H^1(\mathbb{R}^N)$ in a) such that $U|_\Omega = u$ implies automatically that $\|U\|_{H^1(\mathbb{R}^N)} \leq c\|u\|_{H^1(\Omega)}$ for some constant $c > 0$ independent of u.

In fact, let $T : \ H^1(\mathbb{R}^N) \to \widetilde{H}^1(\Omega)$ be defined by $TU := U|_\Omega$. It is clear that $T|_{(\ker T)^\perp} : \ (\ker T)^\perp \to \widetilde{H}^1(\Omega)$ is an isomorphism and hence $(T|_{(\ker T)^\perp})^{-1} : \widetilde{H}^1(\Omega) \to (\ker T)^\perp \subset H^1(\mathbb{R}^N)$ is defined by $(T|_{(\ker T)^\perp})^{-1}u = U$. Therefore

$$\|U\|_{H^1(\mathbb{R}^N)} = \|(T|_{(\ker T)^\perp})^{-1}u\|_{H^1(\mathbb{R}^N)} \leq c\|u\|_{H^1(\Omega)}$$

where $c = \|(T|_{(\ker T)^\perp})^{-1}\|$.

□

Remark 2.3.3. *If $\widetilde{H}^1(\Omega)$ has the extension property in the sense of the preceding definition, then it does not mean that $H^1(\Omega)$ has the extension property. For example, if $\Omega = (0,1) \cup (1,2)$ then $\widetilde{H}^1(\Omega) = H^1(0,2)$ has the extension property but $H^1(\Omega)$ has not since it contains no continuous functions. Let Ω be the unit disk D*

centered at the origine slit along $(-1,0] \times \{0\}$ *in* $\mathbb{R}^2$*. Then* $\widetilde{H}^1(\Omega) = H^1(D)$ *has the extension property; whereas* $H^1(\Omega)$ *does not (see [28, Example 2.1]).*

Note that the converse is always true, since the extension property for $H^1(\Omega)$ *implies that* $\widetilde{H}^1(\Omega) = H^1(\Omega)$*.*

Notice that for the preceding two open sets $\widetilde{H}^1(\Omega)$ *also has the continous extension property.*

Proposition 2.3.4. *Assume that* $\widetilde{H}^1(\Omega)$ *has the continuous extension property. Then* $\mathrm{Cap}(A) = 0$ *if and only if* $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ *for every* $A \subset \bar{\Omega}$*.*

Proof. We show that if $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ then $\mathrm{Cap}(A) = 0$ for every $A \subset \bar{\Omega}$.

First case. Assume that A is a compact set. Let $\varepsilon > 0$. By Proposition 2.2.3 there exists $u \in H^1(\Omega) \cap C_c(\bar{\Omega}) = \widetilde{H}^1(\Omega) \cap C_c(\bar{\Omega})$ such that $u \geq 1$ on A and $\|u\|^2_{H^1(\Omega)} \leq \varepsilon$. Since $\widetilde{H}^1(\Omega)$ has the continuous extension property there exists $U \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ such that $U|_\Omega = u$ and $\|U\|^2_{H^1(\mathbb{R}^N)} \leq c\|u\|^2_{H^1(\Omega)}$. This implies that $U \geq 1$ on A and we obtain that

$$\mathrm{Cap}(A) \leq \|U\|^2_{H^1(\mathbb{R}^N)} \leq c\|u\|^2_{H^1(\Omega)} \leq c\varepsilon.$$

Since ε was arbitrary we conclude that $\mathrm{Cap}(A) = 0$.

Second case. Assume that A is a Borel set. By the property (2.1) for the relative capacity, we have that $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$ for every compact set $K \subset A$. By the first case $\mathrm{Cap}(K) = 0$ for every compact set $K \subset A$ and by the property (2.1) again we obtain that $\mathrm{Cap}(A) = 0$.

Third case. Assume that A is arbitrary. Then by definition of the relative capacity there exists a decreasing sequence of relatively open sets O_n verifying $A \subset O_n$ for every $n \geq 1$ and $\lim_{n\to\infty} \mathrm{Cap}_{\bar{\Omega}}(O_n) = 0$. This implies that $\mathrm{Cap}_{\bar{\Omega}}(\bigcap_{n\geq 1} O_n) = 0$. Since $\bigcap_{n\geq 1} O_n$ is a Borel set, by the second case $\mathrm{Cap}(\bigcap_{n\geq 1} O_n) = 0$. Since $A \subset \bigcap_{n\geq 1} O_n$ it follows that $\mathrm{Cap}(A) = 0$ and the proof is complete. □

Next we give some sufficient conditions on Ω for $\widetilde{H}^1(\Omega)$ to have the continuous extension property. Before we introduce the following class of domain called **Jones domains** (see [63]).

Definition 2.3.5. *Let* $\varepsilon \in (0, \infty)$ *and* $\delta \in (0, \infty]$*. A domain* $D \subset \mathbb{R}^N$ *is called an* (ε, δ)*-domain if whenever* $x, y \in D$ *and* $|x - y| < \delta$*, there is a rectifiable arc* $\gamma \subset D$ *satisfying*

$$l(\gamma) \leq \frac{1}{\varepsilon}|x - y|,$$

and

$$\mathrm{dist}(z, \partial D) \leq \frac{\varepsilon|x - z|\,|y - z|}{|x - y|} \qquad \forall\, z \in \gamma,$$

where $l(\gamma)$ *is the length of* γ*.*

Remark 2.3.6. *a) If Ω is an open subset of $\mathbb{R}$ and $\widetilde{H}^1(\Omega)$ has the extension property then it has the continuous extension property. This follows from the fact that every function $U \in H^1(\mathbb{R})$ is continuous on $\mathbb{R}$.*

b)
- *Assume that $\Omega \subset \mathbb{R}^N$ is an (ε, δ)-domain for some $\varepsilon \in (0, \infty)$ and $\delta \in (0, \infty]$, then $\widetilde{H}^1(\Omega) = H^1(\Omega)$ and by the proof of [63, Theorem 1] $\widetilde{H}^1(\Omega)$ has the continuous extension property.*
- *If $\Omega \subset \mathbb{R}^2$ has the extension property (in the sense of $H^1(\Omega)$) then $\widetilde{H}^1(\Omega) = H^1(\Omega)$ and by [63, Theorem 4] Ω is an (ε, δ)-domain for some $\varepsilon \in (0, \infty)$ and $\delta \in (0, \infty]$. Thus $\widetilde{H}^1(\Omega)$ has the continuous extension property.*
- *Let Ω be an open subset of $\mathbb{R}^2$. If there exists a domain $D \subset \mathbb{R}^2$ satisfying $\Omega \subset D$, $\bar{\Omega} = \bar{D}$, $\widetilde{H}^1(\Omega) = H^1(D)$ and $\widetilde{H}^1(\Omega)$ has the extension property, then by [63, Theorem 4] D is an (ε, δ)-domain for some $\varepsilon \in (0, \infty)$ and $\delta \in (0, \infty]$ and therefore $\widetilde{H}^1(\Omega)$ has the continuous extension property. This is the case of the two domains given in Remark 2.3.3.*

c) In particular, if $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, then $\widetilde{H}^1(\Omega)$ has the continuous extension property (see the proof of [87, Theorem 5' p.184-188]).

The following examples show that if Ω is irregular, there may exist relatively polar subsets of $\partial\Omega$ which are not polar.

First, we note that in the one-dimensional case a set $N \subset \mathbb{R}$ is polar if and only if it is empty. This follows from the fact that each function $u \in H^1(\mathbb{R})$ is continuous on $\mathbb{R}$.

Example 2.3.7. Let $0 < a_{n+1} < b_{n+1} < a_n < 1$ ($n \in \mathbb{N}$) be such that $\lim_{n\to\infty} a_n = 0$, and $\Omega = (0,1) \setminus \bigcup_{n\in\mathbb{N}} [a_n, b_n]$. Then $0 \in \partial\Omega$ and $\mathrm{Cap}_{\bar{\Omega}}(\{0\}) = 0$ whereas $\mathrm{Cap}(\{0\}) > 0$. In fact, the characteristic function $u_n = 1_{[0,a_n]}$ of $[0, a_n]$ is in $\widetilde{H}^1(\Omega)$ and $u_n' = 0$. Since $u_n(x) \geq 1$ on $(0, a_n)$ one has

$$\begin{aligned} \mathrm{Cap}_{\bar{\Omega}}(\{0\}) \leq \|u_n\|^2_{H^1(\Omega)} &= \|u_n\|^2_{L^2(\Omega)} \\ &= \|u_n\|^2_{L^2(0,a_n)} \to 0 \ \ (n \to \infty) \end{aligned}$$

and therefore $\mathrm{Cap}_{\bar{\Omega}}(\{0\}) = 0$. □

Next we modify the 1-dimensional example in order to produce a connected, bounded open set Ω in $\mathbb{R}^2$ and a closed subset of $\partial\Omega$ which is relatively polar but not polar.

Note that by [19, Corollary 5.8.9 p.155], if E is a polar subset of $\mathbb{R}^2$, then E is totally disconnected; that is, every component of E is a singleton.

Example 2.3.8. Let (a_k) and (b_k) be two nonincreasing sequences of real numbers satisfying $a_0, b_0 \in (0,1]$ and $a_k, b_k \to 0$ as $k \to \infty$ and $0 < a_{k+1} < b_{k+1} < a_k < 1$. Define the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} A_k &:= \{(x,y) : a_k < x < b_k, \frac{1}{3} \le y < 1\}, k \ge 0 \\ D &:= \{(x,y) : 0 < x < 1, 0 < y < \frac{1}{3}\} \end{aligned}$$

Let $\Omega := \cup_{k \ge 0} (A_k \cup D)$. Let $f \in C^\infty[0,1]$ be such that

$$f(t) := \begin{cases} 1 & \text{if } t \ge \frac{2}{3} \\ 0 & \text{if } t \le \frac{1}{3}. \end{cases}$$

Let the sequence of functions u_k be defined by:

$$u_k(x,y) := \begin{cases} f(y) & \text{if } x < b_k \\ 0 & \text{elsewhere }. \end{cases}$$

Then $u_k \in H^1(\Omega) \cap C(\bar{\Omega})$, $u_k = 1$ on $F := \{0\} \times [\frac{2}{3}, 1]$ and $0 \le u_k \le 1$. Moreover,

$$\|u_k\|^2_{H^1(\Omega)} = \sum_{m \ge k} (b_m - a_m) \int_{1/3}^1 \left(|f(y)|^2 + |f'(y)|^2\right) \, dy \to 0 \text{ as } k \to \infty.$$

This implies that $\mathrm{Cap}_{\bar{\Omega}}(F) = 0$ but $\mathrm{Cap}(F) > 0$ since $\mathcal{H}^1(F) = 1/3$. □

The domains of Examples 2.3.7 and 2.3.8 have the property that $\mathcal{H}^{N-1}(\partial\Omega) = \infty$. Next we give examples where $\mathcal{H}^{N-1}(\partial\Omega) < \infty$. In the following two examples σ_1 denotes the 1-dimensional and σ_2 the 2-dimensional Hausdorff measure.

Example 2.3.9. Let $\mathbb{Q} \cap (0,1) = \{q_1, q_2, \ldots\}$ where $\mathbb{Q}$ denotes the set of rational numbers. It is clear that $\mathbb{Q} \cap (0,1)$ is dense in $[0,1]$. Consider the following Figure 2.1.

Let $\Omega := \bigcup_{n=1}^\infty \Omega_n$ where $\Omega_n = \bigcup_{i=1}^n \Omega_{n,i}$ as in the Figure 2.1. We assume that the breadth of each rectangle $\Omega_{n,i}$ is 2^{-2n}. Then Ω is an open bounded subset of $\mathbb{R}^2$ but it is not connected. Since $\mathbb{Q} \cap (0,1)$ is dense in $[0,1]$, we have $E := \{0\} \times [0,1] \subset \partial\Omega$ and

$$\partial\Omega \subset \cup_{n=1}^\infty \partial\Omega_n \cup E.$$

Thus $\sigma_1(\partial\Omega) \le 1 + \sum_{n=1}^\infty \sigma_1(\partial\Omega_n)$. Since the 1-dimensional Hausdorff measure of a segment is its length, we have

$$\sigma_1(\partial\Omega_n) \le n(2^{-(n-1)} + 2^{-(2n-1)}).$$

Hence

$$\sigma_1(\partial\Omega) \le 1 + \sum_{n=1}^\infty n(2^{-(n-1)} + 2^{-(2n-1)}) < \infty.$$

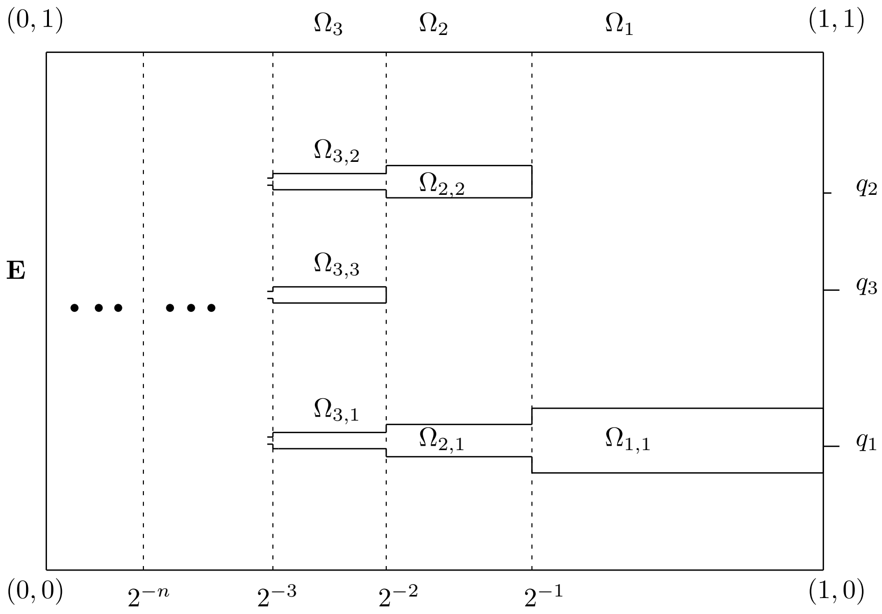

Figure 2.1: Fractal type set

Let $\rho \in C^\infty[0, \infty)$ be such that

$$\begin{cases} 0 \le \rho(x) \le 1 \\ \rho(x) = 1 \quad \text{if} \ \ 0 \le x \le \frac{1}{2} \\ \rho(x) = 0 \quad \text{if} \ \ x > \frac{3}{4}. \end{cases}$$

We define the sequence of functions u_n on Ω by setting $u_n(x, y) := \rho(2^n x)$. Then $u_n \in H^1(\Omega) \cap C(\bar{\Omega})$ and $0 \le u_n(x, y) \le 1$. Since $u_n(0, y) = \rho(0) = 1$, this implies that $u_n = 1$ on E. Moreover,

$$\lim_{n\to\infty} u_n(x, y) = \lim_{n\to\infty} \rho(2^n x) = 0.$$

Since $|u_n(x, y)| = |\rho(2^n x)| \le 1$, Lebesgue's Dominated Convergence Theorem implies that the sequence u_n converges to 0 in $L^2(\Omega)$. Furthermore, $\operatorname{supp}[\nabla u_n] \subset$

$\{X:\ 2^{-n-1} \le x \le 2^{-n}\}$ and

$$\begin{aligned}\int_\Omega |\nabla u_n|^2\, dxdy &= 2^{2n}\int_\Omega |\rho'(2^n x)|^2\, dxdy \qquad (2.16)\\ &\le 2^{2n}(n+1)2^{-2(n+1)}\int_{2^{-(n+1)}}^{2^{-n}} |\rho'(2^n x)|^2\, dx\\ &= (n+1)2^{-(n+2)}\int_{1/2}^{1} |\rho'(r)|^2\, dr\\ &\longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

This implies that $\mathrm{Cap}_{\bar\Omega}(E) = 0$ but clearly, $\sigma_1(E) = 1$. Thus $\mathrm{Cap}(E) > 0$. □

Next we modify Example 2.3.9 to obtain a bounded connected open subset of $\mathbb{R}^3$ and a relatively polar subset of the boundary which is not polar.

Example 2.3.10. Let $D \subset \mathbb{R}^3$ be given by $D := \Omega\times(0,1)\bigcup\cup_{n=1}^\infty P_n$ where Ω is the set of Example 2.3.9 and P_n is a tube of radius $r_n = r^{-2n}$ connecting the walls $(2^{-(n+1)}, 2^{-n})\times\{0\}\times(0,1)$ and $(2^{-(n+1)}, 2^{-n})\times\{1\}\times(0,1)$. Thus D is an open, connected, bounded subset of $\mathbb{R}^3$. Since the 2-dimensional Hausdorff measure of a rectangle in $\mathbb{R}^3$ is its surface, we have

$$\begin{aligned}\sigma_2(\partial D) &\le \sigma_2(\partial\Omega\times[0,1]) + \sum_{n=1}^\infty \sigma_2(\partial P_n)\\ &\le \sigma_1(\partial\Omega) + \sum_{n=1}^\infty 2\pi 2^{-2n} < \infty.\end{aligned}$$

Let $u_n(x,y,z) = \rho(2^n x)$. Then u_n converges to 0 in $L^2(\Omega)$ as $n\to\infty$. Moreover, since $\mathrm{supp}[\nabla u_n] \subset \{x:\ 2^{-n-1}\le x\le 2^{-n}\}$, by (2.16) we have,

$$\begin{aligned}\int_D |\nabla u_n|^2\, dxdydz &\le c_1(n+1)2^{-(n+1)} + 2^{2n}\int_{P_{n+1}} |\rho'(2^n x)|^2\, dxdydz\\ &\le c_1(n+1)2^{-(n+2)} + \|\rho'\|_\infty^2 |2^{2n}\pi r_n^2\\ &\longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Thus $\lim_{n\to\infty} u_n = 0$ in $H^1(D)$. Since $u_n \in H^1(D)\cap C(\bar D)$, $0\le u_n\le 1$ and $u_n = 1$ on $\widetilde E := \{0\}\times[0,1]\times[0,1] \subset \partial D$, it follows that $\mathrm{Cap}_{\bar\Omega}(\widetilde E) = 0$ but $\sigma_2(\widetilde E) = 1$. Thus $\mathrm{Cap}(\widetilde E) > 0$. □

2.4 Another Characterization of $H_0^1(\Omega)$.

Let $\Omega\subset\mathbb{R}^N$ be an open set. It is well-known (see [15, Theorem 1.1] or [55, Example 2.3.1]) that

$$H_0^1(\Omega) = \{u\in H^1(\mathbb{R}^N):\ \tilde u = 0 \ \text{ q.e. on } \ \mathbb{R}^N\setminus\Omega\}. \qquad (2.17)$$

The following result gives a characterization of $H_0^1(\Omega)$ in term of the relative capacity.

Theorem 2.4.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then*

$$H_0^1(\Omega) = \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \ r.q.e. \ on \ \ \partial\Omega\}.$$

To prove the theorem, we need the following abstract result which is contained in [55, Theorem 4.4.3].

Theorem 2.4.2. *Let $(a, D(a))$ be a regular Dirichlet form on $L^2(X, m)$ where X and m satisfy (1.3). For each Borel set $B \subset X$, the space*

$$D(a)_B := \{u \in D(a) : \ \tilde{u} = 0 \ q.e. \ on \ \ B\}$$

is a closed subspace of $(D(a), \|\cdot\|_a)$, where the capacity is taken with respect to the regular form $(a, D(a))$.

If B is closed then the restriction of the form a to the domain $D(a)_B$ is a regular Dirichlet form on $L^2(X \setminus B, m)$. In particular, the space

$$C_B(X) := \{u \in D(a) \cap C_c(X) : \ \mathrm{supp}[u] \subset X \setminus B\}$$

is dense in $D(a)_B$.

Proof of Theorem 2.4.1. Let

$$\widetilde{H}_0^1(\Omega) := \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \ \text{r.q.e. on} \ \ \partial\Omega\}.$$

Let $u \in H_0^1(\Omega)$. Then $u \in H^1(\mathbb{R}^N)$ and $\tilde{u} = 0$ q.e. on $\mathbb{R}^N \setminus \Omega$ (where $\tilde{u}$ denotes the q.c. version of u). Since for every $A \subset \bar{\Omega}$, $\mathrm{Cap}_{\bar{\Omega}}(A) \leq \mathrm{Cap}(A)$, it follows that $u \in \widetilde{H}^1(\Omega)$, $\tilde{u}$ is r.q.c. and $\tilde{u} = 0$ r.q.e. on $\partial\Omega$. This implies that $u \in \widetilde{H}_0^1(\Omega)$ and thus $H_0^1(\Omega) \subset \widetilde{H}_0^1(\Omega)$.

To prove the converse inclusion, let

$$D := \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \ u|_{\partial\Omega} = 0\}.$$

Then $D \subset \widetilde{H}^1(\Omega)$. Since $\partial\Omega$ is relatively closed in $\bar{\Omega}$, by Theorem 2.4.2, the closure of D in $\widetilde{H}^1(\Omega)$ is $\widetilde{H}_0^1(\Omega)$. Since $H_0^1(\Omega)$ contains D (see the proof of Proposition 3.2.1 below) and is a closed subspace of $\widetilde{H}^1(\Omega)$, it follows that $\widetilde{H}_0^1(\Omega) \subset H_0^1(\Omega)$ which completes the proof. □

Next we ask the following question. **Is it possible that for an open set Ω we have $\widetilde{H}^1(\Omega) = H_0^1(\Omega)$?** The following results say that this is not possible for bounded sets, but is well possible for unbounded sets.

Proposition 2.4.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Then* $\mathrm{Cap}_{\bar{\Omega}}(\partial\Omega) > 0$.

Proof. Assume that $\mathrm{Cap}_{\bar\Omega}(\partial\Omega) = 0$. Then, by Theorem 2.4.1, the space $H_0^1(\Omega)$ is equal to $\widetilde{H}^1(\Omega)$. Since Ω is bounded, it follows that the constant function $1 \in H^1(\Omega) \cap C(\bar\Omega)$ and we obtain that $1 \in \widetilde{H}^1(\Omega)$. But $1 \notin H_0^1(\Omega)$ which is a contradiction and thus $\mathrm{Cap}_{\bar\Omega}(\partial\Omega) > 0$. $\square$

Theorem 2.4.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then the following assertions are equivalent.*

(i) $\mathrm{Cap}_{\bar\Omega}(\partial\Omega) = 0$.

(ii) $\bar\Omega = \mathbb{R}^N$ *and* $\mathrm{Cap}(\partial\Omega) = 0$.

(iii) $H_0^1(\Omega) = H^1(\mathbb{R}^N)$.

Proof. (i) $\Rightarrow$ (ii). Assume that $\mathrm{Cap}_{\bar\Omega}(\partial\Omega) = 0$. Then by Theorem 2.4.1, $H_0^1(\Omega) = \widetilde{H}^1(\Omega)$ and thus $\widetilde{H}^1(\Omega)$ has the extension property (since $H_0^1(\Omega)$ has the extension property by [41, p.252]). By Theorem 2.3.4, this implies that $\mathrm{Cap}(\partial\Omega) = 0$. Next, since $\mathrm{Cap}(\partial\Omega) = 0$ and $\mathbb{R}^N \setminus \partial\Omega$ is an open subset of $\mathbb{R}^N$, it follows from [7, Proposition 3.10] that $\mathbb{R}^N \setminus \partial\Omega$ is connected and thus $\bar\Omega = \mathbb{R}^N$.

(ii) $\Rightarrow$ (iii). Assume that $\bar\Omega = \mathbb{R}^N$ and $\mathrm{Cap}(\partial\Omega) = 0$. Then $\mathbb{R}^N \setminus \Omega = \bar\Omega \setminus \Omega = \partial\Omega$ and this implies that $\mathrm{Cap}(\mathbb{R}^N \setminus \Omega) = 0$. By the characterization of $H_0^1(\Omega)$ given by (2.17), this implies that $H_0^1(\Omega) = H^1(\mathbb{R}^N)$.

(iii) $\Rightarrow$ (i). Since we assume that $H_0^1(\Omega) = H^1(\mathbb{R}^N)$, it follows that $\mathrm{Cap}(\mathbb{R}^N \setminus \Omega) = 0$. As $\bar\Omega \subset \mathbb{R}^N$ and $\partial\Omega := \bar\Omega \setminus \Omega \subset \mathbb{R}^N \setminus \Omega$, we obtain that $\mathrm{Cap}(\partial\Omega) = 0$ and therefore by (2.13) $\mathrm{Cap}_{\bar\Omega}(\partial\Omega) = 0$ which completes the proof. $\square$

2.5 Comments.

Section 2.1.
The classical capacity has been introduced in [2], [23], [43], [53], [55], [69], [73], [75] and of course many other authors. We can find a proof of Theorems 2.1.2 and 2.1.3 in [2], [43], [53] and [55]. The proof of Theorem 2.1.5 given here is taken from [43, Theorem 4 p.156].

Section 2.2.
We have defined the relative capacity with the Sobolev space $\widetilde{H}^1(\Omega)$. Here we have used the form $(a_N, \widetilde{H}^1(\Omega))$ defined by

$$a_N(u,v) = \int_\Omega \nabla u \nabla v \, dx$$

which is a regular Dirichlet form on $\bar\Omega$.

In general, without any geometric condition on Ω, the form $(a_N, H^1(\Omega))$ is not a regular Dirichlet form on $\bar\Omega$ since $H^1(\Omega) \cap C_c(\bar\Omega)$ is not always dense in $H^1(\Omega)$. The domain Ω of Remark 2.3.3 is an example, but for this domain, $\partial\Omega \neq \partial\bar\Omega$. For domains in $\mathbb{R}^N$ where $\partial\Omega = \partial\bar\Omega$ and $\widetilde{H}^1(\Omega) \neq H^1(\Omega)$ we refer to [73, Section 1.1.6].

Chen [29] has proved that there exists a regularizing space $\widetilde{\Omega}$ containing Ω as a dense open subset such that $(a_N, H^1(\Omega))$ is a regular Dirichlet form on $\widetilde{\Omega}$. Such $\widetilde{\Omega}$ is obtained by compactification. Moreover, each domain Ω may have many regularizing spaces and if Ω is regular (for example Lipschitz) then one may take $\widetilde{\Omega}$ to be $\bar{\Omega}$. Note that the regularizing space $\widetilde{\Omega}$ is not always a subset of $\mathbb{R}^N$.

It is possible to define a notion of capacity with respect to every Dirichlet form. We have used regular Dirichlet forms because we need some kind of regularity of functions in the Dirichlet space. For example the fact that every function has a quasi-continuous version (Theorem 2.1.3) is true if the capacity is defined with respect to a regular Dirichlet form or (weaker) a quasi-regular Dirichlet form (see [69]). The notion of regular measures with respect to a given capacity which we will introduce in Chapter 3, is also true for a capacity defined with respect to a regular Dirichlet form or (weaker) a quasi-regular Dirichlet form.

The notion of relative capacity will be used to obtain a necessary and sufficient condition for the closability of a class of bilinear forms which we will define in Chapter 3.

Section 2.3.
By Proposition 2.3.4 and Theorem 2.1.5, if $\widetilde{H}^1(\Omega)$ has the continuous extension property, then the restriction σ of $\mathcal{H}^{N-1}$ to $\partial\Omega$ does not charge relatively polar sets in $\mathcal{B}(\partial\Omega)$. Examples 2.3.8, 2.3.9 and 2.3.10 say that the measure σ charges sometimes relatively polar Borel subsets of $\partial\Omega$. In particular, this says that relatively polar subsets of $\partial\Omega$ are not always polar. We shall see in Chapter 4 that for some subsets Ω of $\mathbb{R}^N$ as in Examples 2.3.9 and 2.3.10, we can find some sequence of functions $u_n \in H^1(\Omega) \cap C(\bar{\Omega})$ such that u_n converges to zero in $\widetilde{H}^1(\Omega)$, $u_n|_{\partial\Omega}$ is a Cauchy sequence in $L^2(\partial\Omega, \sigma)$ but $u_n|_{\partial\Omega}$ converges to some function $h \in L^2(\partial\Omega, \sigma)$ which is not zero σ a.e.

Example 2.3.8 will also be used in Chapter 4 to prove that the embedding $\widetilde{H}^1(\Omega) \hookrightarrow L^2(\Omega)$ is not always compact.

Section 2.4.
Another proof of Theorem 2.4.1 is contained in [17]. Since $H^1_0(\Omega)$ can be characterized in terms of the relative capacity, it is also possible to give a characterization in terms of the Hausdorff measure for some regular sets. If Ω is regular (for example Lipschitz), since each function $u \in \widetilde{H}^1(\Omega) = H^1(\Omega)$ has a trace $u|_{\partial\Omega} \in L^2(\partial\Omega, \sigma)$ it is easy to see that $u \in H^1_0(\Omega)$ if and and only if $u \in \widetilde{H}^1(\Omega)$ and $u|_{\partial\Omega} = 0$ σ-a.e. on $\partial\Omega$. Therefore, in that case, assuming that u is r.q.c., we have that $u = 0$ r.q.e. on $\partial\Omega$ is equivalent to $u|_{\partial\Omega} = 0$ σ-a.e. on $\partial\Omega$.

Chapter 3

General Boundary Conditions for the Laplacian

Let $\Omega \subset \mathbb{R}^N$ be an open set. We will define a realization of the Laplacian on $L^2(\Omega)$ with boundary conditions containing the cases Dirichlet, Neumann and Robin. Throughout this chapter the underlying field is $\mathbb{R}$.

3.1 Presentation of the Problem.

Let μ be a Borel measure on $\partial\Omega$ and let

$$E := \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \int_{\partial\Omega} |u|^2 \, d\mu < \infty\}.$$

Define the bilinear symmetric form a_μ with domain E on $L^2(\Omega)$ by

$$a_\mu(u, v) := \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} uv \, d\mu. \tag{3.1}$$

It is natural to ask when (a_μ, E) is closable on $L^2(\Omega)$? Let $u_n \in E$ be such that $u_n \to 0$ in $L^2(\Omega)$ as $n \to \infty$ and $\lim_{n,m\to\infty} a_\mu(u_n - u_m, u_n - u_m) = 0$. Since $u_n \to 0$ in $L^2(\Omega)$ and is a Cauchy sequence in $H^1(\Omega)$, it follows that $u_n \to 0$ in $H^1(\Omega)$ as $n \to \infty$. The following example shows that $u_n|_{\partial\Omega}$ does not always converge to zero in $L^2(\partial\Omega, \mu)$.

Example 3.1.1. Let Ω be a bounded domain in $\mathbb{R}^N$ where $N \geq 2$. Suppose that $H^1(\Omega) \cap C(\bar{\Omega})$ is dense in $H^1(\Omega)$. Fix $z \in \partial\Omega$ and let $\mu := \delta_z$ the Dirac measure at z. Since $H^1(\Omega) \cap C(\bar{\Omega})$ is dense in $H^1(\Omega)$, there exists $u_n \in H^1(\Omega) \cap C(\bar{\Omega})$ such that $u_n \to 0$ in $H^1(\Omega)$ as $n \to \infty$ and $u_n(z) = 1$ for all $n \geq 1$. For a such sequence, we have $\lim_{n,m\to\infty} a_\mu(u_n - u_m, u_n - u_m) = 0$ but $\lim_{n\to\infty} a_\mu(u_n, u_n) = 1$ and thus a_μ is not closable.

Since (a_μ, E) is a symmetric bilinear positive form, by Theorem 1.3.11, there exists a largest closable form $((a_\mu)_r, E)$ which is smaller than (a_μ, E). Using Theorem 1.3.11 and the characterization of closed ideals in L^p-spaces given in Theorem 1.2.4, we will construct the closure of the largest closable part of (a_μ, E) and we will denote its domain by V. Since this closed form $((a_\mu)_r, V)$ is symmetric and densely defined, we can then associate with it a selfadjoint operator Δ_μ on $L^2(\Omega)$ with domain $D(\Delta_\mu^{1/2}) = V$. We will obtain that $\Delta_\mu = \Delta_D$ (the Laplacian with Dirichlet boundary conditions) if μ is *locally infinite* everywhere on $\partial\Omega$. If $\mu = 0$ or if μ is concentrated on a subset B of $\partial\Omega$ such that $\mathrm{Cap}_{\bar{\Omega}}(B) = 0$, then $\Delta_\mu = \Delta_N$ (the Laplacian with Neumann boundary conditions). The case $\mu = \sigma$ (the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure) or more generally, the case where μ is absolutely continuous with respect to σ corresponds to Δ_R (the Laplacian with Robin boundary conditions) which we will study in detail in Chapter 4.

We will also show that in all cases, the operator Δ_μ generates a holomorphic submarkovian C_0-semigroup on $L^2(\Omega)$ which is sandwiched between $(e^{t\Delta_D})_{t\geq 0}$ and $(e^{t\Delta_N})_{t\geq 0}$.

The relative capacity defined in Chapter 2 will play an important role here. We shall prove that we always have the following situation:

1. If μ is locally infinite everywhere, then (a_μ, E) is always closable on $L^2(\Omega)$.
2. If μ is a Radon measure, then (a_μ, E) is closable on $L^2(\Omega)$ if and only if the measure μ does not charge relatively polar Borel subsets of $\partial\Omega$; i.e., μ is absolutely continuous with respect to the relative capacity.

It follows from 1. and 2. that if μ is locally finite on a part of $\partial\Omega$, then (a_μ, E) is closable on $L^2(\Omega)$ if and only if the restriction of μ to the part on which it is locally finite does not charge relatively polar Borel subsets of this part.

3.2 Closable Part by Reed-Simon's Method.

This section is devoted to the construction of the closable part of the form (a_μ, E) defined in Section 3.1. This method of construction has been used by Daners [34] for the case where $\mu = \sigma$. Throughout this section Ω will denote an open subset of $\mathbb{R}^N$.

Proposition 3.2.1. *Let μ be a Borel measure on $\partial\Omega$ and assume that μ is* **locally infinite** *everywhere on $\partial\Omega$; i.e.,*

$$\forall x \in \partial\Omega \ \ \textit{and} \ \ r > 0 \quad \mu(B(x,r) \cap \partial\Omega) = \infty. \tag{3.2}$$

Then the form a_μ is closable and its closure which we denote by a_∞ is given by

$$a_\infty(u,v) = \int_\Omega \nabla u \nabla v \, dx$$

with domain $H_0^1(\Omega)$.

Proof. Let $u \in E$. Since u is continuous on $\bar{\Omega}$, it follows from (3.2) that $u|_{\partial\Omega} = 0$ and thus $E = \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$. One obtains that for all $u, v \in E$,

$$a_\infty(u,v) := a_\mu(u,v) = \int_\Omega \nabla u \nabla v \, dx.$$

It is clear that (a_∞, E) is closable on $L^2(\Omega)$.

Next we prove that $E \subset H_0^1(\Omega)$. Let $u \in E$ and fix a function $G \in C^1(\mathbb{R})$ such that

$$|G(t)| \le |t| \; \forall \, t \in \mathbb{R} \text{ and } G(t) := \begin{cases} 0 & \text{if } |t| \le 1 \\ t & \text{if } |t| \ge 2. \end{cases}$$

Set $u_n := \frac{1}{n} G(nu)$. Then $|u_n(x)| \le \frac{1}{n}|nu(x)| = |u(x)| \in L^2(\Omega)$. Similarly,

$$|D_i u_n(x)| \le |G'(nu(x))| \, |D_i u| \le |D_i u| \in L^2(\Omega).$$

Thus $u_n \in H^1(\Omega)$. It follows from the Lebesgue Dominated Convergence Theorem that $u_n \to u$ in $H^1(\Omega)$. Moreover $\operatorname{supp}[u_n] \subset \{x \in \Omega : \; |u(x)| \ge \frac{2}{n}\}$ which is a compact subset of Ω. Then $u_n \in H_0^1(\Omega)$ and we obtain that $u \in H_0^1(\Omega)$. Since $E \subset H_0^1(\Omega)$ and it contains $\mathcal{D}(\Omega)$, it is dense in $H_0^1(\Omega)$ and it follows that the closure of (a_∞, E) is given by

$$a_\infty(u,v) = \int_\Omega \nabla u \nabla v \, dx$$

with domain $H_0^1(\Omega)$ which corresponds to the form of the Laplacian with Dirichlet boundary conditions. □

Example 3.2.2. Assume that the measure $\mu = 0$. Then $E = H^1(\Omega) \cap C_c(\bar{\Omega})$. Thus the closure of (a_0, E) is given by

$$a_0(u,v) = \int_\Omega \nabla u \nabla v \, dx$$

with domain $\widetilde{H}^1(\Omega)$ which corresponds to the form of the Laplacian with Neumann boundary conditions.

It is also possible to have a measure μ which is locally finite only on a subset of $\partial\Omega$. In this case, we set

$$\Gamma_\infty := \{z \in \partial\Omega : \; \mu(B(z,r) \cap \partial\Omega) = \infty \; \forall \, r > 0\}.$$

Note that Γ_∞ is a relatively closed subset of $\partial\Omega$. As above, $u|_{\Gamma_\infty} = 0$ for each function $u \in E$. Since $\Gamma := \partial\Omega \setminus \Gamma_\infty$ is a locally compact metric space, it follows from [85, Theorem 2.18 p.48] that $\mu|_\Gamma$ (the restriction of μ to Γ) is automatically a regular Borel measure. Therefore μ is a Radon measure on Γ. Without any restriction we may assume that if μ is not locally infinite everywhere on $\partial\Omega$, then it is Radon measure on $\partial\Omega$; i.e. $\Gamma = \partial\Omega$.

In the following, we proceed as in the construction of the closable part in Theorem 1.3.11. Let $L : H^1(\Omega) \to L^2(\Omega) \oplus L^2(\Omega)^N$ be defined by $L(u) := (u, \nabla u)$. It is clear that L is an isometry. We then identify the first order Sobolev space $H^1(\Omega)$ with $L^2(\Omega) \oplus L^2(\Omega)^N$ by the isometry L. Let

$$E_0 := \{(u, \nabla u, u|_{\partial\Omega}) : u \in E\} \subset L^2(\Omega) \oplus L^2(\Omega)^N \oplus L^2(\partial\Omega, \mu).$$

Let $\widetilde{V}$ be the closure of E_0 in the Hilbert space $L^2(\Omega) \oplus L^2(\Omega)^N \oplus L^2(\partial\Omega, \mu)$. Then $\widetilde{V}$ is a Hilbert space and for $(u, v, h) \in \widetilde{V}$, we have $u \in H^1(\Omega)$ and $\nabla u = v$. Let i be the natural embedding from E into $L^2(\Omega)$ and $j : \widetilde{V} \to L^2(\Omega)$ its continuous extension. We know that $(u, v, h) \in \widetilde{V}$ if and only if there exists $u_n \in E \subset H^1(\Omega) \cap C_c(\bar{\Omega})$ satisfying: $u_n \to u$ in $H^1(\Omega)$, $\nabla u = v$ and $u_n|_{\partial\Omega} \to h$ in $L^2(\partial\Omega, \mu)$. Since $(u, v, h) \in \widetilde{V}$ if and only if $u \in H^1(\Omega)$ and $\nabla u = v$, it follows that $\ker j = \{(0, 0, h) \in \widetilde{V}\}$. Let

$$F := \{h \in L^2(\partial\Omega, \mu) : \ (0, 0, h) \in \widetilde{V}\}.$$

Proposition 3.2.3. *There exists a measurable set $S \subset \partial\Omega$ such that*

$$F = \{h \in L^2(\partial\Omega, \mu) : \ h = 0 \ \mu\text{-a.e. on } S\}. \tag{3.3}$$

The proof uses the following two lemmas. Let

$$C_0(\partial\Omega) := \{b \in C(\partial\Omega) : \lim_{|x| \to \infty} b(x) = 0\}.$$

Lemma 3.2.4. *We have $C_0(\partial\Omega)F \subset F$.*

Proof. 1) Let $\psi \in C_c^1(\mathbb{R}^N)$, then $\psi|_{\partial\Omega} h \in F$ for every $h \in F$. In fact, let $h \in F$. Then there exists $u_n \in H^1(\Omega) \cap C_c(\bar{\Omega})$ such that $u_n \to 0$ in $H^1(\Omega)$ and $u_n|_{\partial\Omega} \to h$ in $L^2(\partial\Omega, \mu)$. It is clear that $\psi u_n \in H^1(\Omega) \cap C_c(\bar{\Omega})$. Moreover,

$$\begin{aligned} \|\psi u_n\|^2_{H^1(\Omega)} &= \|\psi u_n\|_2^2 + \|\nabla(\psi u_n)\|_2^2 \\ &\le \|\psi\|_\infty^2 \|u_n\|^2_{H^1(\Omega)} + \|\nabla\psi\|^2_\infty \|u_n\|_2^2 \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

We also have

$$\begin{aligned} \|(\psi u_n)|_{\partial\Omega} - \psi|_{\partial\Omega} h\|_{L^2(\partial\Omega,\mu)} &\le \|\psi|_{\partial\Omega}\|_\infty \|u_n - h\|_{L^2(\partial\Omega,\mu)} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $\psi|_{\partial\Omega} h \in F$.

2) By the Stone-Weierstrass Theorem, the space $\{\Psi|_{\partial\Omega}, \ \Psi \in C_c^1(\mathbb{R}^N)\}$ is dense in $C_0(\partial\Omega)$. We obtain the lemma by passing to the limit. $\square$

Lemma 3.2.5. *Let G be a closed subspace of $L^2(\partial\Omega, \mu)$ such that $C_0(\partial\Omega)G \subset G$. Then $L^\infty(\partial\Omega, \mu)G \subset G$.*

Proof. Let $b \in L^\infty(\partial\Omega, \mu)$ and $v \in G$. We want to prove that $bv \in G$. For $k \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ let

$$A_k := \{x \in \partial\Omega : \ |v(x)| \geq \frac{1}{k}\} \text{ and } A_0 := \{x \in \partial\Omega : \ |v(x)| \neq 0\}.$$

Then, $A_0 = \bigcup_{k\geq 1} A_k$. As $v \in L^2(\partial\Omega, \mu)$, it follows that $\mu(A_k) < \infty$. By Lusin's Theorem ([43, Theorem 1, p.15]), for every $k \in \mathbb{N}^*$ there exists a compact set $C_k \subset A_k$ such that $\mu(A_k \setminus C_k) < \frac{1}{k}$ and $b|_{C_k}$ is continuous. By Tietze's Theorem [43, Theorem 1 p.13], there exists $g_k \in C_0(\partial\Omega)$ such that $g_k|_{C_k} = b$ and

$$\|g_k\|_\infty \leq \|b\|_\infty. \tag{3.4}$$

We can suppose that $C_k \subset C_{k+1}$ (since $A_k \subset A_{k+1}$). Now we prove that $g_k \to b$ μ a.e. on A_0. Let

$$B_n := \{x \in A_0 : \ |g_k(x) - b(x)| \leq \frac{1}{n} \ \text{ for almost all } \ k\}$$

(for "almost all k " signifies for all k except a finite number of k). Then $B_{n+1} \subset B_n$ and $\bigcap_{n\in\mathbb{N}} B_n = \{x \in A_0 : \ \lim_{k\to\infty} g_k(x) = b(x)\}$. Suppose that $(\bigcap_{n\in\mathbb{N}^*} B_n)^c = \bigcup_{n\in\mathbb{N}^*} B_n^c$ is of positive measure. Then there exists $n \in \mathbb{N}^*$ such that $\mu(B_n^c) > 0$. Set $\varepsilon = \frac{1}{n}$, then

$$C := B_n^c = \{x \in A_0 : \ |g_k(x) - b(x)| > \varepsilon \text{ for an infinite number of } k\}$$

satisfies $\mu(C) > 0$. As $A_k \uparrow A_0$, there exist $k_0 \in \mathbb{N}^*, \delta > 0$ such that $\mu(C \cap A_{k_0}) \geq \delta$. For all $k \geq k_0$ we have

$$\mu(C \cap A_k) = \mu(C \cap C_k) + \mu(C \cap (A_k \setminus C_k)) \leq \mu(C \cap C_k) + \frac{1}{k}$$

which implies that for all $k \geq k_0$,

$$\mu(C \cap C_k) \geq \mu(C \cap A_k) - \frac{1}{k} \geq \delta - \frac{1}{k}$$

and therefore

$$\mu(C \cap C_k) > 0$$

for k large enough. Since $b = g_k$ on C_k this is a contradiction. It follows that $g_k \to b$ μ-a.e. on $\partial\Omega$. Since by Lemma 3.2.4 $C_0(\partial\Omega)G \subset G$, it follows that $g_k v \in G$. Using the inequality (3.4) we obtain that $|g_k v|^2 \leq \|b\|_\infty |v|^2$. It follows from Lebesgue's Dominated Convergence Theorem that $g_k v \to bv$ in $L^2(\partial\Omega, \mu)$. Since G is closed in $L^2(\partial\Omega, \mu)$, we have that $bv \in G$. $\square$

Proof of Proposition 3.2.3. It follows from Lemma 3.2.5 that F is a closed ideal of $L^2(\partial\Omega,\mu)$. In fact, let $u \in F$ and $v \in L^2(\partial\Omega,\mu)$ satisfying

$$0 \leq |v| \leq |u|. \tag{3.5}$$

We have to show that $v \in F$. The inequality (3.5) implies that $\frac{v}{u} \in L^\infty(\partial\Omega,\mu)$. Since $v = \frac{v}{u}u$ and by hypothesis $\frac{v}{u} \in L^\infty(\partial\Omega,\mu)$, this implies that $v \in F$.

Finally, since μ is a Radon measure which is σ-finite, the result is a consequence of Theorem 1.2.4. $\square$

Next we define the following subspace of $H^1(\Omega)$

$$V := \{u \in H^1(\Omega) : \exists\, h \in L^2(S,\mu) :\ (u, \nabla u, h) \in \widetilde{V}\}$$

where

$$L^2(S,\mu) := \{h \in L^2(\partial\Omega,\mu) : h = 0\ \mu\text{-a.e. on } S^c\}.$$

It is clear that $H^1_0(\Omega) \subset V$, and thus V is a dense subspace of $L^2(\Omega)$. It is also a Hilbert space for the norm

$$\|u\|_V := \left(\|u\|^2_{H^1(\Omega)} + \int_S |h|^2\, d\mu\right)^{1/2}.$$

Proposition 3.2.6. *For $u \in V$, the function $h \in L^2(S,\mu)$ is unique.*

Proof. Let $h_1, h_2 \in L^2(S,\mu)$ be such that $(u,\nabla u, h_1), (u, \nabla u, h_2) \in \widetilde{V}$. Then $(0,0,h_1 - h_2) \in \widetilde{V}$ which implies that $h_1 - h_2 \in F$ and thus $(h_1 - h_2)|_S = 0$ μ a.e. But $h_1 = h_2 = 0$ μ a.e. on S^c and therefore $h_1 = h_2$ μ a.e. $\square$

Definition 3.2.7. *We call h **the trace of** u and we note $u|_S := h$.*

Lemma 3.2.8. *The following assertions are satisfied.*

a) If $u \in V$ then $u^+ \in V$ and the mapping $u \longmapsto u^+ :\ V \to V$ is continuous.

b) If $u \in V_+$ then $(u-1)^+ \in V_+$ and the mapping $u \longmapsto (u-1)^+ :\ V_+ \to V_+$ is continuous.

c) If $u \in V$ then $u^+|_S = (u|_S)^+$.

d) If $u \in V_+$ then $(u-1)^+|_S = (u|_S - 1)^+$.

Proof. The Sobolev space $H^1(\Omega)$ is a lattice and satisfies the properties a) and b) (see Comments). The space E is also a sublattice of $H^1(\Omega)$ and is dense in V and satisfies the properties c) and d). We obtain the lemma by passing to the limit. $\square$

Next, let the bilinear form which we denote by $((a_\mu)_r, V)$ be defined by

$$(a_\mu)_r(u,v) = \int_\Omega \nabla u \nabla v \, dx + \int_S uv \, d\mu.$$

It is clear that $((a_\mu)_r, V)$ is closed on $L^2(\Omega)$. We will denote by Δ_μ the selfadjoint operator on $L^2(\Omega)$ associated with $((a_\mu)_r, V)$; i.e.,

$$\begin{cases} D(\Delta_\mu) := \{u \in V : \exists\, v \in L^2(\Omega) : (a_\mu)_r(u,\varphi) = (v,\varphi) \;\forall\, \varphi \in V\} \\ \Delta_\mu u := -v. \end{cases} \tag{3.6}$$

Since for each $u \in D(\Delta_\mu)$ we have

$$\int_\Omega \nabla u \nabla \varphi \, dx + \int_S u\varphi \, d\mu = \int_\Omega v\varphi \, dx \tag{3.7}$$

for all $\varphi \in V$, if we choose $\varphi \in \mathcal{D}(\Omega)$, the equality (3.7) can be written

$$\langle -\Delta u, \varphi \rangle = \langle v, \varphi \rangle$$

where $\langle , \rangle$ denotes the duality between $\mathcal{D}(\Omega)'$ and $\mathcal{D}(\Omega)$. Since $\varphi \in \mathcal{D}(\Omega)$ is arbitrary, it follows that

$$-\Delta u = v \quad \text{in } \; \mathcal{D}(\Omega)'.$$

Thus Δ_μ is a realization of the Laplacian on $L^2(\Omega)$. By the Reed-Simon construction (Theorem 1.3.11), $((a_\mu)_r, V)$ is the closure of the closable part of (a_μ, E). Since Ω is arbitrary (without any regularity assumption), we need some further results to characterize the notion of trace given in Definition 3.2.7. We shall give this characterization in Section 3.3.

3.3 Relative Capacity and Closability.

Let $\Omega \subset \mathbb{R}^N$ be an open set. By the results of Section 3.2, we know that if μ is locally infinite everywhere on $\partial\Omega$, then (a_μ, E) is always closable on $L^2(\Omega)$ and its closure is the form of the Laplacian with Dirichlet boundary conditions. We can have a problem of closability of (a_μ, E) if μ is locally finite on $\partial\Omega$ or locally finite only on a part $\Gamma \subset \partial\Omega$. Let

$$\mathcal{M}_0 := \{\mu : \text{ Borel measure on } \partial\Omega : \; \mathrm{Cap}_{\bar\Omega}(N) = 0 \Rightarrow \mu(N) = 0 \;\forall\, N \in \mathcal{B}(\partial\Omega)\}.$$

Theorem 3.3.1. *Let μ be a Radon measure on $\partial\Omega$. Then the following assertions are equivalent.*

(i) The form (a_μ, E) is closable on $L^2(\Omega)$.

(ii) $\mu \in \mathcal{M}_0$.

Proof. (ii) $\Rightarrow$ (i). The condition that μ is locally finite is not necessary for this part. Let $u_k \in E$ be such that $u_k \to 0$ in $L^2(\Omega)$ and $\lim_{n,k\to\infty} a_\mu(u_n-u_k, u_n-u_k) = 0$. It is clear that $u_k \to 0$ in $H^1(\Omega)$. By Theorem 2.1.3 applied to the relative capacity, the sequence (u_k) contains a subsequence (w_k) which converges to zero r.q.e. on $\bar{\Omega}$. Since $\mu \in \mathcal{M}_0$, it follows that $w_k|_{\partial\Omega} \to 0$ μ a.e. Without lost the generality, we may assume that $u_k|_{\partial\Omega} \to 0$ μ a.e. Since u_k is a Cauchy sequence in $L^2(\partial\Omega,\mu)$, it follows that $u_k \to 0$ in $L^2(\partial\Omega,\mu)$ and thus the form (a_μ, E) is closable on $L^2(\Omega)$.

(i) $\Rightarrow$ (ii). Assume that there exists a Borel set $K \subset \partial\Omega$ such that $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$ and $\mu(K) > 0$. We may assume that K is a compact set. Since $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$, by Theorem 2.2.4, there exists a sequence $u_k \in H^1(\Omega) \cap C_c(\bar{\Omega})$ such that

$$0 \le u_k \le 1,\ u_k = 1 \text{ on } K \text{ and } \|u_k\|_{H^1(\Omega)} \to 0 \text{ as } k \to \infty.$$

Let (A_i) be a sequence of relatively open sets with compact closure satisfying

$$K \subset \overline{A_{i+1}} \subset A_i \subset \partial\Omega,\ \bigcap_i \overline{A_i} = K \text{ and } \mu(A_i) \to \mu(K) \text{ as } i \to \infty.$$

There then exists a sequence $v_i \in \mathcal{D}(\mathbb{R}^N)$ such that $\mathrm{supp}[v_i] \subset A_i,\ v_i = 1$ on K and $0 \le v_i \le 1$. Clearly, $v_i|_\Omega \in H^1(\Omega) \cap C_c(\bar{\Omega})$ and $\|u_k v_i\|_{H^1(\Omega)} \to 0$ as $k \to \infty$. For all $i \ge 1$ we have $u_k v_i \in H^1(\Omega) \cap C_c(\bar{\Omega})$. Moreover, for all $i,k,\ 0 \le u_k v_i \le 1$ and $u_k v_i = 1$ on K. For all $i \ge 1$, we choose $k_i \in \mathbb{N}$ such that $\|u_{k_i} v_i\|_{H^1(\Omega)} \le \frac{1}{2^i}$. Let $w_i = u_{k_i} v_i$. Then $w_i \to 0$ in $H^1(\Omega)$ as $i \to \infty$, $0 \le w_i \le 1$ and $w_i = 1$ on K. Moreover $w_i \to \chi_K$ pointwise since $\mathrm{supp}[w_i] \subset A_i$. We choose I large enough such that for $i > I$ we have $|w_i(x)|^2 \le \varepsilon$ for all $x \in \partial\Omega \setminus K$. Then for $i,j > I$ we obtain

$$\begin{aligned}
\int_{\partial\Omega} |w_i - w_j|^2\, d\mu &= \int_K |w_i - w_j|^2\, d\mu + \int_{\partial\Omega\setminus K} |w_i - w_j|^2\, d\mu \\
&= \int_{\partial\Omega\setminus K} |w_i - w_j|^2\, d\mu \\
&\le 2\varepsilon.
\end{aligned}$$

Thus w_i is a Cauchy sequence in $L^2(\partial\Omega,\mu)$. Since $w_i = 1$ on K it follows that

$$\|w_i\|^2_{L^2(\partial\Omega,\mu)} \ge \mu(K) > 0.$$

The existence of a such sequence contradicts the closability of a_μ. $\square$

Corollary 3.3.2. *Let μ be a Borel measure on $\partial\Omega$ and let*

$$\Gamma := \{z \in \partial\Omega :\ \exists\, r > 0 :\ \mu(B(z,r) \cap \partial\Omega) < \infty\}$$

be the relatively open subset of $\partial\Omega$ on which μ is locally finite. Then the following assertions are equivalent.

(i) The form (a_μ, E) is closable on $L^2(\Omega)$.

(ii) The measure μ does not charge relatively polar Borel subsets of Γ.

It follows from the preceding corollary that the class $\mathcal{M}_0$ can be defined

$$\mathcal{M}_0 = \{\mu : \text{ Borel measure on } \partial\Omega : \ \mathrm{Cap}_{\bar\Omega}(N) = 0 \Rightarrow \mu(N) = 0 \ \forall\, N \in \mathcal{B}(\Gamma)\}$$

where Γ denotes the relatively open subset of $\partial\Omega$ on which μ is locally finite.

Example 3.3.3. a) Consider the form (a_μ, E) defined in Example 3.1.1 by

$$a_\mu(u,v) := \int_\Omega \nabla u \nabla v \, dx + u(z)v(z), \qquad z \in \partial\Omega.$$

We have shown that this form is not closable on $L^2(\Omega)$. Since $\mathrm{Cap}_{\bar\Omega}(\{z\}) = 0$ and $\mu(\{z\}) > 0$, it follows that $\mu \notin \mathcal{M}_0$ and by Theorem 3.3.1, (a_μ, E) is not closable on $L^2(\Omega)$.

b) Assume that $\Omega \subset \mathbb{R}$ is a bounded domain; i.e., $\Omega = (a,b)$. Then for every $x \in [a,b]$, we have $\mathrm{Cap}_{\bar\Omega}(\{x\}) > 0$. This follows from the fact that $\widetilde{H}^1(a,b) = H^1(a,b) \hookrightarrow C[a,b]$. We consider the form (a_μ, E) defined by

$$a_\mu(u,v) := \int_a^b u'v' \, dx + u(a)v(a)$$

with $E = H^1(a,b) \cap C[a,b] = H^1(a,b)$. Since $\mu \in \mathcal{M}_0$, it follows that (a_μ, E) is closable on $L^2(\Omega)$.

Let μ be a Borel measure on $\partial\Omega$ in $\mathcal{M}_0$. By definition, the domain V of the closure of the form (a_μ, E) is the completion of E with respect to the a_μ-norm. The following result gives a characterization of V. Before, note that, throughout the following, for $u \in \widetilde{H}^1(\Omega)$, we will always choose the r.q.c. version $\tilde u$ which is Borel measurable.

Proposition 3.3.4. *Let μ be a Borel measure on $\partial\Omega$ in $\mathcal{M}_0$. Then*

$$V = \{u \in \widetilde{H}^1(\Omega) : \ \tilde u \in L^2(\partial\Omega, \mu)\} \tag{3.8}$$

where $\tilde u$ denotes the r.q.c. version of u.

Proof. Let

$$W := \{u \in \widetilde{H}^1(\Omega) : \ \tilde u \in L^2(\partial\Omega, \mu)\}.$$

Recall that

$$E = \{u \in H^1(\Omega) \cap C_c(\bar\Omega) : \int_{\partial\Omega} |u|^2 \, d\mu < \infty\}.$$

It suffices to prove that E is a_μ-dense in W. Let Γ_∞ be the relatively closed subset of $\partial\Omega$ on which μ is locally infinite. Set $\Gamma := \partial\Omega \setminus \Gamma_\infty$ and $X := \Omega \cup \Gamma$. Then X is relatively open and

$$H^1(\Omega) \cap C_c(X) \subset E \subset \{u \in \widetilde{H}^1(\Omega) : \ \tilde u = 0 \ \text{ r.q.e. on } \ \Gamma_\infty\} := \widetilde{E}$$

where

$$C_c(X) := \{u \in C(\bar{\Omega}) : \ \mathrm{supp}[u] \subset X\}.$$

Note that, in general, $C_c(X)$ is defined differently. If μ is a Radon measure on $\partial\Omega$ then $\Gamma_\infty = \emptyset$. Notice that $\widetilde{E}$ is a closed subspace of $\widetilde{H}^1(\Omega)$ (see Theorem 2.4.2).

a) We claim that W is dense in $\widetilde{E}$. Indeed, since X is relatively open, it follows from Theorem 2.4.2 that $H^1(\Omega) \cap C_c(X)$ is dense in $\widetilde{E}$. Since $H^1(\Omega) \cap C_c(X) \subset W \subset \widetilde{E}$, the claim is proved.

b) Let Γ_k be an increasing sequence of relatively open subsets of Γ such that $\overline{\Gamma_k} \subset \Gamma$ and $\bigcup_k \Gamma_k = \Gamma$ and let $X_k := \Omega \cup \Gamma_k$. Then X_k is relatively open for each $k \in \mathbb{N}$. Let

$$\begin{aligned} \widetilde{E}_k &:= \{u \in \widetilde{E} : \ \tilde{u} = 0 \ \text{ r.q.e. on } \ \Gamma \setminus \Gamma_k\} \\ &= \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \text{ r.q.e. on } \ \partial\Omega \setminus \Gamma_k\}. \end{aligned}$$

Since $H^1(\Omega) \cap C_c(X) \subset \bigcup_k \widetilde{E}_k$, it follows that $\bigcup_k \widetilde{E}_k$ is dense in $\widetilde{E}$. Since X_k is relatively open, by Theorem 2.4.2, the space

$$H^1(\Omega) \cap C_c(X_k) := \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \ \mathrm{supp}[u] \subset X_k\}$$

is dense in $\widetilde{E}_k$. Let

$$\begin{aligned} \widetilde{E}_k^\mu &:= \{u \in \widetilde{E}_k : \ \tilde{u} \in L^2(\Gamma_k, \mu)\} \\ &= \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \text{ r.q.e. on } \ \partial\Omega \setminus \Gamma_k : \ \tilde{u} \in L^2(\Gamma_k, \mu)\} \end{aligned}$$

be equipped with the a_μ-norm. We claim that $H^1(\Omega) \cap C_c(X_k)$ is a_μ-dense in $\widetilde{E}_k^\mu$. Indeed, let $u \in \widetilde{E}_k^\mu$. Without restriction, we assume that u is r.q.c. By considering u^+ and u^- separetely and making truncations if necessary, we may assume that $0 \le u \le \gamma$ for some constant γ. Since $u \in \widetilde{E}_k$, it follows from the above that there exists a sequence $u_n \in H^1(\Omega) \cap C_c(X_k)$ such that $u_n \to u$ in $\widetilde{E}_k$ (i.e. in $\widetilde{H}^1(\Omega)$). Put $u_n^{(\gamma)} := (0 \vee u_n) \wedge \gamma$ for each n. Clearly $u_n^{(\gamma)} \in H^1(\Omega) \cap C_c(X_k)$ and it converges to u in $\widetilde{H}^1(\Omega)$ as $n \to \infty$. By Theorem 2.1.3 applied to the relative capacity, there exists a subsequence which we also denote by $u_n^{(\gamma)}$ such that $u_n^{(\gamma)} \to u$ r.q.e. and in particular μ a.e. (since $\mu \in \mathcal{M}_0$). Since $\mu(\Gamma_k) < \infty$, we have that $u_n^{(\gamma)} \to u$ in $L^2(\Gamma_k, \mu)$ by the Lebesgue Dominated Convergence Theorem proving that $u_n^{(\gamma)} \to u$ in $\widetilde{E}_k^\mu$ and the claim is proved.

c) Let

$$\begin{aligned} W_k &:= \{u \in W : \ \tilde{u} = 0 \ \text{ r.q.e. on } \ \partial\Omega \setminus \Gamma_k\} \\ &= \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \text{ r.q.e. on } \ \partial\Omega \setminus \Gamma_k : \ \tilde{u} \in L^2(\Gamma_k, \mu)\}. \end{aligned}$$

It is clear that for each $k \in \mathbb{N}$ we have $W_k = \widetilde{E}_k^\mu$. We claim that $\bigcup_k W_k = \bigcup_k \widetilde{E}_k^\mu$ is dense in W. Indeed, let $u \in W$ and suppose $u \ge 0$ a.e. so that $\tilde{u} \ge 0$ r.q.e.

Let $u_n \in \bigcup_k \widetilde{E}_k$ be a sequence which converges to u in $\widetilde{E}$. Let $v_n := (u \wedge u_n) \vee 0$. Then $v_n \to u$ in $\widetilde{E}$. Taking a subsequence we may assume that $\tilde{v}_n \to \tilde{u}$ r.q.e. Thus $\tilde{v}_n \to \tilde{u}$ μ-a.e. and also $0 \le \tilde{v}_n \le \tilde{u}$ μ-a.e. so that $\tilde{v}_n \to \tilde{u}$ in $L^2(\Gamma, \mu)$ by the Lebesgue Dominated Convergence Theorem. Since v_n already converges to u in $\widetilde{E}$, we have that $v_n \to u$ in W. Moreover, $v_n = 0$ whenever $u_n = 0$ and so $v_n \in W_k$ for some $k \in \mathbb{N}$. For arbitrary $u \in W$, we apply this argument separetely to the positive and negative parts, u^+, u^- which completes the proof of the claim.

d) Now, since $H^1(\Omega) \cap C_c(X) = \bigcup_k H^1(\Omega) \cap C_c(X_k)$ and $W_k = \widetilde{E}_k^\mu$, it follows from b) and c) that $H^1(\Omega) \cap C_c(X)$ is dense in W and therefore E is dense in W and the proof is complete. $\square$

It follows from the preceding proposition that for a given Borel measure on $\partial\Omega$ in $\mathcal{M}_0$, the closed form (a_μ, V) is given by

$$a_\mu(u, v) = \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\mu,$$

where V is given by (3.8).

Next we give the following decomposition of Radon measures.

Lemma 3.3.5. *Let $\mathcal{R}$ be the set of all Radon measures on $\partial\Omega$. For each $\mu \in \mathcal{R}$ there exists a unique pair (μ_r, μ_s) of measures on $(\partial\Omega, \mathcal{B}(\partial\Omega))$ such that*

a) $\mu = \mu_r + \mu_s$.

b) $\mu_r(A) = 0$ *for every* $A \in \mathcal{B}(\partial\Omega)$ *with* $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$.

c) $\mu_s = \chi_N \mu$ *for some* $N \in \mathcal{B}(\partial\Omega)$ *with* $\mathrm{Cap}_{\bar{\Omega}}(N) = 0$.

Proof. 1) We prove the uniqueness of the decomposition. Assume that there exist $\tilde{\mu}_r$ and $\tilde{\mu}_s$ such that $\mu = \tilde{\mu}_r + \tilde{\mu}_s$ with $\tilde{\mu}_s = \chi_{\tilde{N}} \mu$ for some relatively polar Borel set $\tilde{N}$. Then $\mu_r - \tilde{\mu}_r = \tilde{\mu}_s - \mu_s$. Let $M := N \cup \tilde{N}$. Then M is a relatively polar Borel set. By definition, for every $A \in \mathcal{B}(\partial\Omega)$, we have that

$$(\mu_r - \tilde{\mu}_r)(A) = (\tilde{\mu}_s - \mu_s)(A) = (\tilde{\mu}_s - \mu_s)(A \cap M) = (\mu_r - \tilde{\mu}_r)(A \cap M) = 0.$$

Thus $\mu_r(A) = \tilde{\mu}_r(A)$ and $\mu_s(A) = \tilde{\mu}_s(A)$ for every $A \in \mathcal{B}(\partial\Omega)$ with completes the proof of the uniqueness.

2) We see the existence of μ_r and μ_s.

(i) Let $K \subset \partial\Omega$ be a compact set. We show that there exists $S \subset K$ such that $\chi_S \mu \in \mathcal{M}_0$ and $\mathrm{Cap}_{\bar{\Omega}}(K \setminus S) = 0$. Since $\mu(K) < \infty$, it follows that

$$\alpha := \sup\{\mu(A) : A \in \mathcal{B}(\partial\Omega), A \subset K \text{ and } \mathrm{Cap}_{\bar{\Omega}}(A) = 0\} < \infty.$$

Take an increasing sequence $A_n \in \mathcal{B}(\partial\Omega)$ with $A_n \subset K$ such that $\mathrm{Cap}_{\bar{\Omega}}(A_n) = 0$ for all n and $\lim_{n\to\infty} \mu(A_n) = \alpha$. Let $N := \cup_{n=1}^\infty A_n$. Then $N \in \mathcal{B}(\partial\Omega)$, $N \subset K$,

$$\mathrm{Cap}_{\bar{\Omega}}(N) \le \sum_{n=1}^\infty \mathrm{Cap}_{\bar{\Omega}}(A_n) = 0 \Rightarrow \mathrm{Cap}_{\bar{\Omega}}(N) = 0$$

and
$$\mu(N) = \mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n) = \alpha.$$
We show that $\mu(B \setminus N) = 0$ if $B \subset K$ is a Borel set such that $\mathrm{Cap}_{\bar{\Omega}}(B) = 0$. Assume that $\mu(B \setminus N) > 0$. Let $A = B \cup N$. Then $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ but $\mu(A) = \mu(B \setminus N) + \mu(N) > \alpha$ contradicting the definition of α. Now let $S = K \setminus N$. Then $\chi_S \mu \in \mathcal{M}_0$ and $\mathrm{Cap}_{\bar{\Omega}}(K \setminus S) = 0$.

(ii) Let $K_n \subset K_{n+1}$ be compact sets such that $K_n \subset K_{n+1}$ and $\bigcup_{n\in\mathbb{N}} K_n = \partial\Omega$. By (i) there exist sets $S_n \subset K_n$ such that $\chi_{S_n}\mu \in \mathcal{M}_0$ and $\mathrm{Cap}_{\bar{\Omega}}(K_n \setminus S_n) = 0$. Let $S := \bigcup_{n\in\mathbb{N}} S_n$ and $N := \partial\Omega \setminus S$. Then
$$\mu(A \setminus N) = 0 \text{ for every } A \in \mathcal{B}(\partial\Omega) \text{ with } \mathrm{Cap}_{\bar{\Omega}}(A) = 0. \tag{3.9}$$
We then define μ_r, μ_s by
$$\mu_r := \chi_S \mu \text{ and } \mu_s := \chi_N \mu.$$
Obviously, (μ_r, μ_s) enjoys the properties a) and c). Moreover, (3.9) implies that b) is satisfied. □

Definition 3.3.6. *We call the measure μ_r the* **regular part** *of μ with respect to the relative capacity.*

Remark 3.3.7. *If μ is not a Radon measure on $\partial\Omega$, since its restriction to the part Γ on which it is locally finite is a Radon measure, we can also decompose $\mu|_\Gamma = \mu_r + \mu_s$ as in Lemma 3.3.5. For simplicity, we assume throughout the following that $\Gamma = \partial\Omega$.*

Proposition 3.3.8. *Let μ be a Radon measure on $\partial\Omega$ and $\mu = \mu_r + \mu_s$ be its decomposition as in Lemma 3.3.5. Then the closure (a_{μ_r}, V) of the closable part of (a_μ, E) is given by*
$$\begin{aligned} a_{\mu_r}(u,v) &= \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\mu_r \\ &= \int_\Omega \nabla u \nabla v \, dx + \int_S \tilde{u}\tilde{v} \, d\mu \end{aligned} \tag{3.10}$$
with domain
$$V = \{u \in \tilde{H}^1(\Omega) : \ \tilde{u} \in L^2(\partial\Omega, \mu_r)\}$$
where $S := \partial\Omega \setminus N$.

Proof. a) We show that the form (a_{μ_r}, V) is closed on $L^2(\Omega)$. It suffices to prove that $(V, \|\cdot\|_{a_{\mu_r}})$ is a Hilbert space. Let $u_n \in V$ be a a_{μ_r}-Cauchy sequence. We assume that the u_n are relatively quasi-continuous. Then
$$\lim_{n,m\to\infty} \|u_n - u_m\|_{H^1(\Omega)} + \lim_{n,m\to\infty} \|u_n - u_m\|_{L^2(\partial\Omega,\mu_r)} = 0.$$

Since u_n is a Cauchy sequence in $\widetilde{H}^1(\Omega)$, it converges to an element $u \in \widetilde{H}^1(\Omega)$. By Theorem 2.1.3 b), after taking a subsequence if necessary, we may assume that u is r.q.c. and u_n converges to u r.q.e. on $\bar{\Omega}$. Since $\mu_r \in \mathcal{M}_0$, this implies that $u_n|_{\partial\Omega}$ converges to $u|_{\partial\Omega}$ μ_r-a.e. Since $u_n|_{\partial\Omega}$ is a Cauchy sequence in $L^2(\partial\Omega, \mu_r)$, it converges to an element $h \in L^2(\partial\Omega, \mu_r)$. After a subsequence, we may assume that $u_n|_{\partial\Omega}$ converges to h μ_r-a.e. Now the uniqueness of the limit implies that $h = u|_{\partial\Omega}$ μ_r-a.e. Therefore $u|_{\partial\Omega} \in L^2(\partial\Omega, \mu_r)$ and thus $(V, \|\cdot\|_{a_{\mu_r}})$ is complete.

b) Let $(a_r, D(a_r))$ be the closure of the closable part of a_μ. We have to prove that $a_r = a_{\mu_r}$. Since a_{μ_r} is closed (by a)) and $a_{\mu_r} \le a_\mu$, it follows that $a_{\mu_r} \le a_r$. Let us prove that $a_r \le a_{\mu_r}$. Since $\mu_s := \chi_N \cdot \mu$ is a regular Borel measure, there exists an increasing sequence of compact sets $K_n \subset N$ such that $\mu_s(N \setminus \cup_n K_n) = 0$. Since N is relatively polar, it follows that $\mathrm{Cap}_{\bar{\Omega}}(K_n) = 0$ and as K_n is a compact set, there exists a sequence $\psi_n \in H^1(\Omega) \cap C_c(\bar{\Omega})$ such that:

$$0 \le \psi_n \le 1, \; \psi_n = 1 \text{ on } K_n, \; \psi_n \to 0 \text{ in } H^1(\Omega) \text{ as } n \to \infty.$$

(i) We show that $a_r(\varphi, \varphi) \le a_{\mu_r}(\varphi, \varphi)$ for every $\varphi \in E$ which is dense in V and in $D(a_r)$. Let $\varphi \in E$ and $\varphi_n = (1 - \psi_n)\varphi$. We claim that $\varphi_n \to \varphi$ in $H^1(\Omega)$. In fact,

$$\begin{aligned}\int_\Omega |\varphi_n - \varphi|^2 \, dx &= \int_\Omega |\varphi\psi_n|^2 \, dx \\ &\le \|\varphi\|_\infty^2 \int_\Omega |\psi_n|^2 \, dx \to 0 \quad (n \to \infty).\end{aligned}$$

Thus $\varphi_n \to \varphi$ in $L^2(\Omega)$. Moreover, it is clear that $D_j\varphi_n = (1 - \psi_n)D_j\varphi - (D_j\psi_n)\varphi \to D_j\varphi$ $(n \to \infty)$ in $L^2(\Omega)$ since $\psi_n \to 0$ a.e. in Ω and $D_j\psi_n \to 0$ in $L^2(\Omega)$. We have shown that $\varphi_n \to \varphi$ in $H^1(\Omega)$. By Theorem 2.1.3 b), the sequence φ_n contains a subsequence which we also denote by φ_n which converges r.q.e. on $\bar{\Omega}$. Since $\mu_r \in \mathcal{M}_0$, we have that $\varphi_n|_{\partial\Omega} \to \varphi|_{\partial\Omega}$ μ_r a.e. Since $|\varphi_n| \le |\varphi|$, it follows from Lebesgue's Dominated Convergence Theorem that $\varphi_n|_{\partial\Omega} \to \varphi|_{\partial\Omega}$ in $L^2(\partial\Omega, \mu_r)$. By construction $\varphi_n|_{\partial\Omega} \to 0$ μ_s a.e. and by Lebesgue's Dominated Convergence Theorem again,

$$\int_N |\varphi_n|^2 \, d\mu \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Thus we obtain that

$$\lim_{n,m\to\infty} a_\mu(\varphi_n - \varphi_m, \varphi_n - \varphi_m) = 0.$$

Using the fact that the form a_r is closed and (3.11) we obtain that

$$a_r(\varphi, \varphi) = \lim_{n\to\infty} a_r(\varphi_n, \varphi_n) \le \lim_{n\to\infty} a_\mu(\varphi_n, \varphi_n) = \lim_{n\to\infty} a_{\mu_r}(\varphi_n, \varphi_n) = a_{\mu_r}(\varphi, \varphi),$$

where in the last equality, we use the fact that a_{μ_r} is closed.

(ii) We show that $a_r \le a_{\mu_r}$; i.e., $V \subset D(a_r)$ and $a_r(\varphi,\varphi) \le a_{\mu_r}(\varphi,\varphi)$ for every $\varphi \in V$. Let $\varphi \in V$. There then exists a sequence $\varphi_n \in E$ such that $\varphi_n \to \varphi$ in V. It follows from (i) that $a_r(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \le a_{\mu_r}(\varphi_n - \varphi_m, \varphi_n - \varphi_m)$ and this implies that φ_n is a Cauchy sequence in $D(a_r)$ and then converges to an element $\psi \in D(a_r)$. Since $\varphi_n \to \varphi$ in $L^2(\Omega)$, the uniqueness of the limit implies that $\psi = \varphi \in D(a_r)$ and thus $V \subset D(a_r)$. Finally, since $a_r(\varphi_n, \varphi_n) \le a_{\mu_r}(\varphi_n, \varphi_n)$, taking the limit as $n \to \infty$ we obtain that $a_r(\varphi,\varphi) \le a_{\mu_r}(\varphi,\varphi)$ for all $\varphi \in V$. Thus $a_r \le a_{\mu_r}$ which completes the proof. □

Remark 3.3.9. *Since the closed form $((a_\mu)_r, V_r)$ constructed in Section 3.2 is by construction the closure of the closable part of the form (a_μ, E), it follows from the preceding proposition that $((a_\mu)_r, V_r) = (a_{\mu_r}, V)$ and therefore the set $S \subset \partial\Omega$ of Lemma 3.2.5 is exactly $\partial\Omega \setminus N$. In fact, recall that*

$$V_r = \{u \in H^1(\Omega) :\ \exists\, h \in L^2(S,\mu) :\ (u, \nabla u, h) \in \widetilde{V}\}$$

where

$$L^2(S,\mu) := \{h \in L^2(\partial\Omega,\mu) :\ h = 0\ \mu \text{ a.e. on } \partial\Omega \setminus S\},$$

and $\widetilde{V}$ is defined in Section 3.2. **Proof.** Let $u \in V$. There then exists a sequence $u_n \in E$ such that u_n converges to u in $\widetilde{H}^1(\Omega)$ and $u_n|_{\partial\Omega}$ converges to $\tilde{u}$ in $L^2(\partial\Omega, \mu_r)$ where $\tilde{u}$ denotes the relatively quasi-continuous version of u. As in the proof of Proposition 3.3.8, the sequence u_n can be chosen such that $\int_{\partial\Omega\setminus S} |u_n|^2\, d\mu$ converges to zero as $n \to \infty$. Now let $h := \tilde{u}\chi_S$. Then $h \in L^2(S,\mu)$ and $u_n|_{\partial\Omega}$ converges to h in $L^2(\partial\Omega,\mu)$. We have shown that $V \subset V_r$.

To prove the converse inclusion, let $u \in V_r$. By definition, there exists $h \in L^2(S,\mu)$ such that $(u, \nabla u, h) \in \widetilde{V}$. This means that, there exists a sequence $u_n \in E$ such that $u_n \to u$ in $H^1(\Omega)$ and $u_n|_{\partial\Omega} \to h$ in $L^2(\partial\Omega, \mu)$. Since

$$\int_{\partial\Omega} |u_n - h|^2\, dy = \int_S |u_n - h|^2\, d\mu + \int_{\partial\Omega\setminus S} |u_n|^2\, d\mu;$$

which converges to zero as $n \to \infty$, it follows that

$$\int_S |u_n - h|^2\, d\mu = \int_{\partial\Omega} |u_n - h|^2\, d\mu_r$$

converges to zero as $n \to \infty$. Proceeding as the proof of the preceding proposition we obtain that $u_n|_{\partial\Omega} \to \tilde{u}|_{\partial\Omega}$ μ_r-a.e. and the uniqueness of the limit implies that $h = \tilde{u}|_{\partial\Omega}$ μ_r-a.e. We have shown that $V_r \subset V$. □

It then follows that for every $u \in V$, the trace h of u defined in Definition 3.2.7 is in fact $\chi_S\tilde{u}$.

Example 3.3.10. Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a bounded domain and let

$$\Gamma := \{z_n \in \partial\Omega :\ n \in \mathbb{N}\},\ z_n \ne z_m \text{ for } n \ne m \text{ and } \mu := \sum_{n\in\mathbb{N}} \frac{1}{n^2}\delta_{z_n}$$

where we assume that Γ is dense in $\partial\Omega$ and δ_z denotes the Dirac measure at z. Since $N \geq 2$, for each $n \in \mathbb{N}$ we have $\mathrm{Cap}_{\bar{\Omega}}(\{z_n\}) = 0$ and thus $\mathrm{Cap}_{\bar{\Omega}}(\Gamma) = 0$. But $\mu(\{z_n\}) > 0$. We obtain that the regular part μ_r of μ is the measure 0 and a_0 defined by

$$a_0(u,v) = \int_\Omega \nabla u \nabla v \, dx$$

with domain $\widetilde{H}^1(\Omega)$ is the closure of the closable part of (a_μ, E).

Now, let Δ_μ be the selfadjoint operator on $L^2(\Omega)$ associated with the closure (a_μ, V) of the closable part of (a_μ, E) as defined in (3.6). It follows from Theorem 1.3.4 that the operator Δ_μ generates a holomorphic C_0-semigroup $T_\mu = (e^{t\Delta_\mu})_{t\geq 0}$ on $L^2(\Omega)$. In the following sections, we shall give some properties of this semigroup T_μ.

3.4 Some Properties of T_μ.

Throughout this section, $(e^{t\Delta_D})_{t\geq 0}$ (resp. $(e^{t\Delta_N})_{t\geq 0}$) will denote the submarkovian semigroups on $L^2(\Omega)$ generated by Δ_D (resp. Δ_N). We shall prove in this section that $e^{t\Delta_\mu}$ is sandwiched between $e^{t\Delta_N}$ and $e^{t\Delta_D}$. Moreover, under some additional conditions, each symmetric semigroup sandwiched between $e^{t\Delta_D}$ and $e^{t\Delta_N}$ is of the form $e^{t\Delta_\mu}$.

Theorem 3.4.1. *Let μ be a Borel measure on $\partial\Omega$ in $\mathcal{M}_0$ and Δ_μ be the closed operator on $L^2(\Omega)$ associated with the closure of the form (a_μ, E). Then*

$$0 \leq e^{t\Delta_D} \leq e^{t\Delta_\mu} \leq e^{t\Delta_N}$$

for all $t \geq 0$ in the sense of positive operators.

To prove this result, we need the following result characterizing domination of positive semigroups due to Ouhabaz and contained in [79, Théorème 3.1.7].

Theorem 3.4.2 (Ouhabaz). *Let T and S be two positive symmetric C_0-semigroups on $L^2(\Omega)$. Let $(a, D(a))$ be the closed form associated with T and $(b, D(b))$ the closed form associated with S. Then the following assertions are equivalent.*

(i) $T(t) \leq S(t)$ for all $t \geq 0$ in the sense of positive operators.

(ii) $D(a)$ is an ideal of $D(b)$ and $b(u,v) \leq a(u,v)$ for all $u, v \in D(a)_+$.

Proof of Theorem 3.4.1. 1) We prove the inequality $e^{t\Delta_D} \leq e^{t\Delta_\mu}$. Recall that the forms associated to Δ_D and Δ_μ are given respectively by

$$a_D(u,v) = \int_\Omega \nabla u \nabla v \, dx, \quad u, v \in H_0^1(\Omega)$$

and

$$a_\mu(u,v) = \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v}, \quad u,v \in V$$

where

$$V = \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} \in L^2(\partial\Omega, \mu)\}.$$

By Theorem 3.4.2, it suffices to prove that $H_0^1(\Omega)$ is an ideal of V and $a_\mu(u,v) \le a_D(u,v)$ for all $u, v \in H_0^1(\Omega)_+$. We may assume that each $u \in \widetilde{H}^1(\Omega)$ is r.q.c.

a) We claim that $H_0^1(\Omega)$ is an ideal of V. In fact, let $u \in H_0^1(\Omega)$ and $v \in V$ be such that $0 \le |v| \le |u|$. Since $\bar{\Omega}$ is relatively open, it follows from Theorem 2.1.3 c) that $0 \le |v| \le |u|$ r.q.e. on $\bar{\Omega}$. Using the characterization of $H_0^1(\Omega)$ given by Theorem 2.4.1, we obtain that $u = 0$ r.q.e. on $\partial\Omega$ and thus $v = 0$ r.q.e. on $\partial\Omega$. Therefore $v \in H_0^1(\Omega)$ which proves the claim.

b) Let $u, v \in H_0^1(\Omega)_+$. By the characterization of $H_0^1(\Omega)$, we have that $u = v = 0$ r.q.e. on $\partial\Omega$. Since $\mu \in \mathcal{M}_0$, it follows that $u = v = 0$ μ a.e. on $\partial\Omega$. We finally obtain that

$$\begin{aligned} a_\mu(u,v) &= \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} uv \, d\mu \\ &= \int_\Omega \nabla u \nabla v \, dx \\ &= a_D(u,v) \end{aligned}$$

and the proof of this part is complete.

2) The proof of the inequality $e^{t\Delta_\mu} \le e^{t\Delta_N}$ is a simple modification of the first part. □

More generally, we have the following result.

Proposition 3.4.3. *Let μ, ν be two Borel measures on $\partial\Omega$ in $\mathcal{M}_0$. Assume that $\nu \le \mu$ in the sense that $\nu(A) \le \mu(A)$ for all $A \in \mathcal{B}(\partial\Omega)$. Let Δ_ν and Δ_μ denote the selfadjoint operators on $L^2(\Omega)$ associated respectively with the closure of the forms (a_ν, E) and (a_μ, E). Then*

$$0 \le e^{t\Delta_D} \le e^{t\Delta_\mu} \le e^{t\Delta_\nu} \le e^{t\Delta_N}$$

for all $t \ge 0$ in the sense of positive operators.

Proof. 1) We show that $e^{t\Delta_\mu} \le e^{t\Delta_\nu}$ for all $t \ge 0$ in the sense of positive operator. Let

$$E_\nu := \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \int_{\partial\Omega} |u|^2 \, d\nu < \infty\}$$

and

$$E_\mu := \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \int_{\partial\Omega} |u|^2 \, d\mu < \infty\}.$$

Recall that V_ν (resp. V_μ) is the completion of E_ν (resp. E_μ) with respect to the a_ν-norm (resp. a_μ-norm) and by Proposition 3.3.4, they are given by

$$V_\nu := \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} \in L^2(\partial\Omega, \nu)\}$$

and

$$V_\mu := \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} \in L^2(\partial\Omega, \mu)\}.$$

Since $\nu \leq \mu$, it is clear that V_μ is continuously embedded into V_ν.

a) We claim that V_μ is an ideal of V_ν. Let $u \in V_\mu$ and $v \in V_\nu$ be such that $0 \leq |v| \leq |u|$. We have to show that $v \in V_\mu$. We may assume that u and v are r.q.c. It is clear that $v \in \widetilde{H}^1(\Omega)$. Since $0 \leq |v| \leq |u|$ a.e., it follows that $0 \leq |v| \leq |u|$ r.q.e. and therefore μ, ν a.e. (since μ, $\nu \in \mathcal{M}_0$). It then follows that

$$\int_{\partial\Omega} |v|^2 \, d\mu \leq \int_{\partial\Omega} |u|^2 \, d\mu < \infty$$

and therefore $v \in L^2(\partial\Omega, \mu)$ which proves the claim.

b) Let $0 \leq u, v \in V_\mu$. We have that $0 \leq u, v$ r.q.e. on $\bar{\Omega}$ and thus μ a.e. on $\partial\Omega$. Therefore

$$\begin{aligned} a_\nu(u,v) &= \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\nu \\ &\leq \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\mu \\ &= a_\mu(u,v) \end{aligned}$$

which completes the proof.

2) The other inequalities follow from Theorem 3.4.1. $\square$

Remark 3.4.4. *Theorem 3.4.1 implies that $(e^{t\Delta_\mu})_{t\geq 0}$ is a submarkovian semigroup on $L^2(\Omega)$. Then, by Theorem 1.3.17, it induces consistent positive contractive semigroups on $L^p(\Omega)$, $1 \leq p \leq \infty$ which are strongly continuous for $1 \leq p < \infty$.*

Next we ask the following question. Let μ be a Borel measure on $\partial\Omega$ in $\mathcal{M}_0$. **Is the closed form (a_μ, V) always regular on $\bar{\Omega}$?**

Proposition 3.4.5. *Let μ be a Borel measure on $\partial\Omega$ in $\mathcal{M}_0$ and let (a_μ, V) be the closure of the form (a_μ, E). Then the following assertions are equivalent.*

(i) (a_μ, V) is regular on $\bar{\Omega}$.

(ii) μ is a Radon measure.

Proof. (i) $\Rightarrow$ (ii). Assume that (a_μ, V) is regular on $\bar{\Omega}$. Then we can define a Choquet capacity $\mathrm{Cap}^\mu_{\bar{\Omega}}$ on $\bar{\Omega}$ with respect to the form (a_μ, V) as we have defined the relative capacity. Since (a_μ, V) is regular on $\bar{\Omega}$, by a well-known result (see [55, p.6]), for every compact set $K \subset \bar{\Omega}$ and every relatively open set $O \subset \bar{\Omega}$ satisfying $K \subset O \subset \bar{\Omega}$, there exists a function $u \in V \cap C_c(\bar{\Omega})$ such that $u = 1$ on K, $u = 0$ on $\bar{\Omega} \setminus O$ and $0 \leq u \leq 1$. This implies that $\mathrm{Cap}_{\bar{\Omega}}^{\mu}(K) < \infty$ for every compact set $K \subset \bar{\Omega}$. Let $K \subset \partial\Omega$ be an arbitrary compact set. Since for all $u \in V \cap C_c(\bar{\Omega})$ we have

$$\|u\|_{L^2(\partial\Omega,\mu)}^2 \leq \|u\|_V^2,$$

taking the infimum over all functions $u \in V \cap C_c(\bar{\Omega})$ satisfying $u \geq 1$ on K, we obtain that

$$\mu(K) \leq \mathrm{Cap}_{\bar{\Omega}}^{\mu}(K) < \infty$$

and thus μ is a Radon measure on $\partial\Omega$.

(ii) $\Rightarrow$ (i). Assume that μ is a Radon measure. Since $E = H^1(\Omega) \cap C_c(\bar{\Omega}) \subset V \cap C_c(\bar{\Omega})$ and is dense in V, it follows that $V \cap C_c(\bar{\Omega})$ is dense in V and uniformly dense in $C_c(\bar{\Omega})$. □

Corollary 3.4.6. *Assume that Ω is a bounded open set. Let μ be a Borel measure on $\partial\Omega$ in $\mathcal{M}_0$ and let (a_μ, V) be the closure of the form (a_μ, E). Then the following assertions are equivalent.*

(i) $1 \in V$.

(ii) *μ is a finite Borel measure.*

(iii) *(a_μ, V) is regular on $\bar{\Omega}$.*

(iv) *V is dense in $\widetilde{H}^1(\Omega)$.*

Proof. (i) $\Rightarrow$ (ii). Assume that $1 \in V$. Since

$$V = \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} \in L^2(\partial\Omega, \mu)\},$$

it follows that $\mu(\partial\Omega) < \infty$ and therefore μ is a finite Borel measure.

(ii) $\Rightarrow$ (i). Assume that μ is a finite Borel measure. Then $1 \in E = H^1(\Omega) \cap C(\bar{\Omega}) \subset V$.

(ii) $\Leftrightarrow$ (iii). This part follows from Proposition 3.4.5.

(ii) $\Rightarrow$ (iv). Assume that μ is finite Borel measure. Then $E = H^1(\Omega) \cap C(\bar{\Omega}) \subset V \subset \widetilde{H}^1(\Omega)$ and V is trivially dense in $\widetilde{H}^1(\Omega)$.

(iv) $\Rightarrow$ (ii). Assume that V is dense in $\widetilde{H}^1(\Omega)$. We claim that E is dense in $\widetilde{H}^1(\Omega)$. Let $\varepsilon > 0$ and $u \in \widetilde{H}^1(\Omega)$. By hypothesis, there exists $v \in V$ such that

$$\|v - u\|_{H^1(\Omega)} < \varepsilon.$$

Since E is dense in V, there exists $w \in E$ such that

$$\|w - v\|_V < \varepsilon.$$

We obtain that

$$\begin{aligned}\|w-u\|_{H^1(\Omega)} &\leq \|w-v\|_{H^1(\Omega)} + \|v-u\|_{H^1(\Omega)} \\ &\leq \|w-v\|_V + \|v-u\|_{H^1(\Omega)} \\ &\leq 2\varepsilon\end{aligned}$$

and thus E is dense in $\widetilde{H}^1(\Omega)$. Assume that μ is not a finite measure. Then $\mu(\partial\Omega) = \infty$ and therefore μ is locally infinite on a relatively closed subset Γ_∞ of $\partial\Omega$. Therefore

$$H^1(\Omega) \cap C_c(\Omega \cup (\partial\Omega \setminus \Gamma_\infty)) \subset E \subset \{u \in H^1(\Omega) \cap C(\bar{\Omega}) : \ u|_{\Gamma_\infty} = 0\}.$$

By Theorem 2.4.2 the closure of E in $\widetilde{H}^1(\Omega)$ is given by

$$\widetilde{E} = \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \text{ r.q.e. on } \ \Gamma_\infty\}.$$

Since E is dense in $\widetilde{H}^1(\Omega)$, it follows that $\mathrm{Cap}_{\bar{\Omega}}(\Gamma_\infty) = 0$ which is a contradiction. Thus $\mu(\partial\Omega) < \infty$ and the proof is complete. □

Remark 3.4.7. *In Corollary 3.4.6, if we drop the hypothesis $\mu \in \mathcal{M}_0$, then we always have the following implications: (i) ⇒ (ii) ⇒ (iii) ⇒ (iv). But (iv) ⇒ (ii) is not always true as the following example shows.*

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and ν be a finite Borel measure on $\partial\Omega$ in $\mathcal{M}_0$. We define a measure μ on $\partial\Omega$ as follows: for $B \in \mathcal{B}(\partial\Omega)$ we let

$$\mu(B) := \begin{cases} \infty & \text{if } \ \mathrm{Cap}_{\bar{\Omega}}(B) = 0, \ B \neq \emptyset \\ \nu(B) & \text{otherwise} . \end{cases}$$

Then μ is a Borel measure on $\partial\Omega$. Let Γ_∞ be the part of $\partial\Omega$ on which μ is locally infinite. By the proof of Proposition 3.4.6, the closure of E in $\widetilde{H}^1(\Omega)$ is $\widetilde{E}$. Since $\mathrm{Cap}_{\bar{\Omega}}(\Gamma_\infty) = 0$, it follows that $\widetilde{E} = \widetilde{H}^1(\Omega)$ and thus V is dense in $\widetilde{H}^1(\Omega)$ but μ is not a finite measure on $\partial\Omega$.

By Proposition 3.4.5, for a given Borel measure μ on $\partial\Omega$ in $\mathcal{M}_0$, the closed form (a_μ, V) is not always regular on $\bar{\Omega}$. The following result shows that it is always regular on some relatively open subset of $\bar{\Omega}$.

Proposition 3.4.8. *Let μ be a Borel measure on $\partial\Omega$ in $\mathcal{M}_0$ and (a_μ, V) be the closure of the form (a_μ, E). Then there exists a relatively open set X satisfying $\Omega \subset X \subset \bar{\Omega}$ such that the Dirichlet form (a_μ, V) is regular on X.*

Proof. Let

$$X := \bar{\Omega} \setminus \{x \in \bar{\Omega} : \ u(x) = 0 \ \forall\, u \in V \cap C_c(\bar{\Omega})\}.$$

Then X is relatively open in $\bar{\Omega}$. Since

$$\{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \ u|_{\partial\Omega} = 0\} \subset V \cap C_c(\bar{\Omega}) \subset H^1(\Omega) \cap C(\bar{\Omega}),$$

it follows that $\Omega \subset X \subset \bar{\Omega}$. We show that (a_μ, V) is regular on X.

a) If μ is a Radon measure, then $X = \bar{\Omega}$ and by Proposition 3.4.5 (a_μ, V) is regular on $X = \bar{\Omega}$.

b) If μ is not a Radon measure, let Γ_∞ be the relatively closed subset of $\partial\Omega$ on which μ is locally infinite. Then it is clear that

$$\begin{aligned} \Gamma_\infty &:= \{x \in \partial\Omega : \ \mu(B(x,r) = \infty \ \ \forall\, r > 0\} \\ &= \{x \in \bar{\Omega} : \ u(x) = 0 \ \ \forall\, u \in V \cap C(\bar{\Omega})\} \end{aligned}$$

and therefore $X = \bar{\Omega} \setminus \Gamma_\infty$. Recall that by Proposition 3.3.4,

$$V = \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} \in L^2(\partial\Omega, \mu)\}.$$

By the proof of Proposition 3.3.4, the space $H^1(\Omega) \cap C_c(X)$ is a_μ-dense in V. To complete the proof, it suffices to show that $H^1(\Omega) \cap C_c(X)$ is uniformly dense in $C_c(X)$. Since $(a_N, \widetilde{H}^1(\Omega))$ is a regular Dirichlet form on $\bar{\Omega}$ having $H^1(\Omega) \cap C_c(\bar{\Omega})$ as a core and since X is relatively open, it follows from Theorem 2.4.2 that $H^1(\Omega) \cap C_c(X)$ is uniformly dense in $C_c(X)$ (see also Lemma 3.4.22 below) which completes the proof. □

Next we investigate under which conditions two measures in $\mathcal{M}_0$ determine the same semigroup. Before, we need some definitions.

Along with the relatively quasi-continuity of functions, we can introduce the relative quasi notions of subsets of $\bar{\Omega}$.

Definition 3.4.9. *a) A set $O \subset \bar{\Omega}$ is called* **relatively quasi-open** *if for every $\varepsilon > 0$ there exists a relatively open set G_ε containing O with $\mathrm{Cap}_{\bar{\Omega}}(G_\varepsilon \setminus O) < \varepsilon$.*

b) A **relatively quasi-closed** *set is by definition the complement of a relatively quasi-open set.*

Using an abstract result which is contained in [55, Lemma 4.6.1], we can prove the following result.

Lemma 3.4.10. *A set $F \subset \bar{\Omega}$ is relatively quasi-closed if and only if there exists a nonnegative r.q.c. function $u \in \widetilde{H}^1(\Omega)$ with $F = u^{-1}(\{0\})$ r.q.e.; i.e., up to a relatively polar set.*

Definition 3.4.11. *Let μ and ν be two Borel measures on $\partial\Omega$ in $\mathcal{M}_0$. We say that μ* **is equivalent to** *ν ($\mu \sim \nu$) if $e^{t\Delta_\mu} = e^{t\Delta_\nu}$ for all $t \geq 0$.*

Proposition 3.4.12. *Let μ and ν be two Borel measures on $\partial\Omega$ in $\mathcal{M}_0$. Then the following assertions are equivalent.*

(i) $\mu \sim \nu$.

(ii) $\int_{\partial\Omega} |u|^2 \, d\mu = \int_{\partial\Omega} |u|^2 \, d\nu \ \ \forall\, u \in \widetilde{H}^1(\Omega)$, *$u$ r.q.c.*

(iii) $\mu(G) = \nu(G)$ *for each relatively quasi-open subset* G *of* $\partial\Omega$.

Proof. (i) $\Rightarrow$ (ii). Assume that μ is equivalent to ν. Then $V_\mu = V_\nu$ and $a_\mu(u,v) = a_\nu(u,v)$ for all $u, v \in V_\mu = V_\nu$; i.e.,

$$\int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\mu = \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\nu.$$

In particular,

$$\int_{\partial\Omega} |u|^2 \, d\mu = \int_{\partial\Omega} |u|^2 \, d\nu$$

for all r.q.c. $u \in V_\mu = V_\nu$. Let $u \in \widetilde{H}^1(\Omega)$ be r.q.c. but not in $V_\mu = V_\nu$. Then

$$\int_{\partial\Omega} |u|^2 \, d\mu = \infty = \int_{\partial\Omega} |u|^2 \, d\nu$$

and the proof of (ii) is complete.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. It suffices to show that $V_\mu = V_\nu$ and $a_\mu(u,u) = a_\nu(u,u)$ for all $u \in V_\mu$. We may assume that each function $u \in \widetilde{H}^1(\Omega)$ is r.q.c. The condition (ii) implies that

$$\int_{\partial\Omega} |u|^2 \, d\mu = \int_{\partial\Omega} |u|^2 \, d\nu$$

for all $u \in V_\mu$ and for all $u \in V_\nu$. Then we obtain easily that $V_\mu = V_\nu$ and $a_\mu(u,u) = a_\nu(u,u)$ and therefore $\mu \sim \nu$.

(ii) $\Rightarrow$ (iii). Let $G \subset \partial\Omega$ be relatively quasi-open. Then by Lemma 3.4.10, there exists a nonnegative function $u \in \widetilde{H}^1(\Omega)$ which is r.q.c. such that $G = \{u > 0\}$ up to a relatively polar set and then up to a μ, ν null set (since $\mu, \nu \in \mathcal{M}_0$). Let $v_k := (ku) \wedge 1$ for $k \in \mathbb{N}$. Then the functions $v_k \in \widetilde{H}^1(\Omega)$ and are r.q.c. By (ii),

$$\int_{\partial\Omega} |v_k|^2 \, d\mu = \int_{\partial\Omega} |v_k|^2 \, d\nu.$$

Thus

$$\lim_{k\to\infty} \int_{\partial\Omega} |v_k|^2 \, d\mu = \lim_{k\to\infty} \int_{\partial\Omega} |v_k|^2 \, d\nu.$$

It is also clear that $v_k^2 \uparrow \chi_{\{u>0\}}$ as $k \to \infty$. Using the Monotone Convergence Theorem (see [43, Theorem 2, p.20]), we obtain that

$$\int_{\partial\Omega} \lim_{k\to\infty} |v_k|^2 \, d\mu = \int_{\partial\Omega} \lim_{k\to\infty} |v_k|^2 \, d\nu;$$

i.e.,

$$\int_{\{u>0\}} d\mu = \int_{\{u>0\}} d\nu$$

and thus $\mu(G) = \nu(G)$.

(iii) $\Rightarrow$ (ii). Assume that (iii) holds. Let $u \in \widetilde{H}^1(\Omega)$ be r.q.c. By definition, for every $\varepsilon > 0$ there exists G_ε relatively open such that $\mathrm{Cap}_{\bar\Omega}(G_\varepsilon) < \varepsilon$ and $u|_{\bar\Omega \setminus G_\varepsilon}$ is continuous. We may enlarge G_ε to make it relatively quasi-closed without changing its relative capacity. Let φ be an arbitrary nonnegative continuous function on $\mathbb{R}$ and let $f := \varphi o u$. Then f is r.q.c. This implies that the inverse image under f of an open set is r.q.e. quasi-open. So

$$\int_0^\infty \mu(f > t)\, dt = \int_0^\infty \nu(f > t)\, dt$$

or

$$\int_{\partial\Omega} f\, d\mu = \int_{\partial\Omega} f\, d\nu.$$

Taking $\varphi(x) = x^2$, we obtain (ii) and the proof is complete. $\square$

Remark 3.4.13. *If one of the measure μ and ν is a Radon measure, then the condition (iii) of Proposition 3.4.12 implies that $\mu = \nu$.*

Proposition 3.4.14. *Let μ and ν be two Borel measures on $\partial\Omega$ in $\mathcal{M}_0$. Let*

$$\Gamma_\mu := \{z \in \partial\Omega : \exists\, r > 0\ \mu(B(z, \partial\Omega) \cap \partial\Omega) < \infty\}$$

(resp. Γ_ν) denote the part of $\partial\Omega$ on which μ is locally finite (resp. the part of $\partial\Omega$ on which ν is locally finite). Then the following assertions are equivalent.

(i) $\mu \sim \nu$.

(ii) $\Gamma_\mu \cong \Gamma_\nu$ *(i.e.,* $\mathrm{Cap}_{\bar\Omega}(\Gamma_\mu \triangle \Gamma_\nu) = 0$*) and* $\mu = \nu$ *on* $\Gamma := \Gamma_\mu \cap \Gamma_\nu$.

Proof. (i) $\Rightarrow$ (ii). Assume that $\mu \sim \nu$. Let (a_μ, V_μ) (resp. (a_ν, V_ν)) be the closure of the form (a_μ, E_μ) (resp. (a_ν, E_ν)). Then $V_\mu = V_\nu$. We may assume that each function $u \in \widetilde{H}^1(\Omega)$ is r.q.c. Since each function $u \in V_\mu = V_\nu$ satisfies $u = 0$ r.q.e. on Γ_μ^c and in Γ_ν^c, it follows that $\Gamma_\mu^c \cong \Gamma_\nu^c$ and therefore $\Gamma_\mu \cong \Gamma_\nu$. Since Γ is relatively open, it follows from Proposition 3.4.12 that $\mu(\Gamma) = \nu(\Gamma)$. Moreover, for every relatively quasi-open set $O \subset \Gamma$ we have $\mu(O) = \nu(O)$. Since $\mu|_\Gamma$ and $\nu|_\Gamma$ are Radon measures, it follows that $\mu|_\Gamma = \nu|_\Gamma$.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. Let $u \in V_\mu$. We may assume that u is r.q.c. Since μ is locally infinite on Γ^c, we have that $u = 0$ r.q.e. on Γ^c and therefore μ a.e. Since $\mu = \nu$ on Γ, it follows that

$$\int_{\partial\Omega} |u|^2\, d\nu = \int_\Gamma |u|^2\, d\nu = \int_\Gamma |u|^2\, d\mu < \infty$$

and thus $V_\mu \subset V_\nu$. Similarly, we obtain the converse inclusion. It is easy to see that $a_\mu(u, v) = a_\nu(u, v)$ for all $u, v \in V_\mu = V_\nu$ and therefore $\mu \sim \nu$. $\square$

Recall that, by Theorem 3.4.1, for each Borel measure μ on $\partial\Omega$ in $\mathcal{M}_0$, the C_0-semigroup $(e^{t\Delta_\mu})_{t\geq 0}$ on $L^2(\Omega)$ given by μ is always between $(e^{t\Delta_D})_{t\geq 0}$ and $(e^{t\Delta_N})_{t\geq 0}$. A very natural question is the following. **Is the converse also true?** More precisely, if $T = (T(t))_{t\geq 0}$ is a C_0-semigroup on $L^2(\Omega)$ satisfying

$$e^{t\Delta_D} \leq T(t) \leq e^{t\Delta_N} \tag{3.12}$$

for all $t \geq 0$ in the sense of positive operators, is $T(t)$ always given by a measure μ on $\partial\Omega$? The following example shows that this is not always the case.

Example 3.4.15. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Let B be a bounded linear positive operator from $H^1(\Omega)$ into $L^2(\partial\Omega, \sigma)$ where σ denotes the usual Lebesgue surface measure. Consider the bilinear form a_B with domain $H^1(\Omega)$ on $L^2(\Omega)$ defined by

$$a_B(u, v) := \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} (Bu)v \, d\sigma.$$

Since for all $u \in H^1(\Omega)$, we have

$$\|Bu\|_{L^2(\partial\Omega,\sigma)} \leq c_1 \|u\|_{H^1(\Omega)}$$

and

$$\|u\|_{L^2(\partial\Omega,\sigma)} \leq c_2 \|u\|_{H^1(\Omega)}$$

(since Ω has a Lipschitz boundary), it follows that the form a_B is closed on $L^2(\Omega)$. Let Δ_B be the closed operator on $L^2(\Omega)$ associated with the closed form a_B. The operator Δ_B generates a C_0-semigroup on $L^2(\Omega)$ and using Theorem 3.4.2, we obtain easily that $e^{t\Delta_B}$ satisfies (3.12), but it is not given by a measure on $\partial\Omega$ in general.

More precisely, let $\Omega = (0, 1)$. Define the form a with domain $H^1(0, 1)$ by,

$$a(u, v) = \int_0^1 u'v' \, dx + u(0)v(0) + u(1)v(0) + u(0)v(1) + u(1)v(1).$$

Then $(a, H^1(0, 1))$ is a regular Dirichlet form on $[0, 1]$. Let T be the associated semigroup on $L^2(0, 1)$. Then $T(t)$ satisfies (3.12) but the semigroup T is not given by a measure on $\partial\Omega$.

Now we would like to determine under which conditions, each semigroup on $L^2(\Omega)$ between $(e^{t\Delta_D})_{t\geq 0}$ and $(e^{t\Delta_N})_{t\geq 0}$ is given by a measure μ on $\partial\Omega$. For this we need some preparations.

Let X and m satisfying (1.3) and $(a, D(a))$ be a symmetric closed form on $L^2(X, m)$. Let A be the nonnegative selfadjoint operator on $L^2(X, m)$ associated with $(a, D(a))$ and $\{G_\alpha, \ \alpha > 0\}$ the resolvent corresponding to A. It is well-known that $G_\alpha(L^2(X, m)) \subset D(a)$ and if $u \in L^2(X, m)$ and $v \in D(a)$

$$a_\alpha(G_\alpha u, v) = (u, v),$$

where $a_\alpha(u,v) := a(u,v) + \alpha(u,v)$ for $u,v \in D(a)$. We define a symmetric form $a^{(\alpha)}$ on $L^2(X,m)$ by: for $u,v \in L^2(X,m)$ we let

$$a^{(\alpha)}(u,v) := \alpha(u - \alpha G_\alpha u, v). \tag{3.13}$$

Lemma 3.4.16. *For every $u \in L^2(X,m)$, $a^{(\alpha)}(u,u)$ is nondecreasing as $\alpha \uparrow \infty$ and*

$$\begin{cases} D(a) & = \{u \in L^2(X,m) :\ \lim_{\alpha\to\infty} a^{(\alpha)}(u,u) < \infty\} \\ a(u,v) & = \lim_{\alpha\to\infty} a^{(\alpha)}(u,v), \quad u,v \in D(a). \end{cases}$$

Idea of the proof. In fact, the resolvent equation and the contraction property of G_α imply that G_α is nonnegative definite and $(G_\alpha u,u) \leq \frac{1}{\alpha}(u,u)$. Hence $a^{(\alpha)}(u,u) \geq 0$. By the resolvent equation again

$$\begin{cases} \frac{d}{d\alpha} a^{(\alpha)}(u,u) & = (\alpha G_\alpha u - u, \alpha G_\alpha u - u) \geq 0 \\ \frac{d^2}{d\alpha^2} a^{(\alpha)}(u,u) & = -2(v, G_\alpha v) \leq 0 \end{cases}$$

where $v = \alpha G_\alpha u - u$. We see in particular that $a^{(\alpha)}(u,u)$ is nondecreasing as $\alpha \uparrow \infty$. The second part of the lemma can be proved by using the Spectral Theorem (see [55, Section 1.3]) or by a direct computation (see [69, Theorem I.2.13]). $\square$

Lemma 3.4.16 says that $a^{(\alpha)}$ is an approximating form determined by G_α. The following result is contained in [55, Lemma 1.4.1].

Lemma 3.4.17. *If S is a positive symmetric linear operator on $L^2(X,m)$, then there exists a unique positive Radon measure ν on the product space $X \times X$ satisfying the following property: for all $u,v \in L^2(X,m)$,*

$$(u, Sv) = \int_{X\times X} u(x)v(y)\, d\nu.$$

Next, for a measurable function u on X we let

$$\mathrm{supp}[u] := \mathrm{supp}[|u| \cdot m] \tag{3.14}$$

and call $\mathrm{supp}[u]$ **the support** of u. Note that if $u = v$ m-a.e., then $\mathrm{supp}[u] = \mathrm{supp}[v]$. Hence $\mathrm{supp}[u]$ is well-defined by (3.14) for all $u \in L^2(X,m)$. If $u \in C(X)$, then $\mathrm{supp}[u]$ is just the closure of $\{x \in X :\ u(x) \neq 0\}$.

Definition 3.4.18. *Let $(a, D(a))$ be a symmetric Dirichlet form on $L^2(X,m)$. We say that the form $(a, D(a))$ is* **local** *if $a(u,v) = 0$ for all $u,v \in D(a)$ with disjoint compact supports.*

The following result shows that under the regularity asumption of the form $(a, D(a))$, "$\mathrm{supp}[u]$, $\mathrm{supp}[v]$ compact" in the definition of local forms can be dropped and gives another property which is equivalent with the local property. The first part of the proof is contained in [69, Proposition V.1.2] and the second part in [69, p.150].

Proposition 3.4.19. *Assume that $(a, D(a))$ is a symmetric regular Dirichlet form on $L^2(X, m)$. Then the following assertions are equivalent.*

(i) $(a, D(a))$ *is local.*

(ii) $a(u, v) = 0$ *for all* $u, v \in D(a)$ *with* $\operatorname{supp}[u] \cap \operatorname{supp}[v] = \emptyset$.

(iii) $a(u, v) = 0$ *for all* $u, v \in D(a) \cap C_c(X)$ *with* $\operatorname{supp}[u] \cap \operatorname{supp}[v] = \emptyset$.

Now we return to the form (a_μ, V).

Proposition 3.4.20. *Let μ be a Borel measure on $\partial\Omega$ in $\mathcal{M}_0$ and (a_μ, V) be the closure of the form (a_μ, E). Then the form (a_μ, V) is local.*

Proof. Let Γ_∞ be the part of $\partial\Omega$ on which μ is locally infinite and let $X := \Omega \cup (\partial\Omega \setminus \Gamma_\infty)$. Since by Proposition 3.4.8, the form (a_μ, V) is regular on X and has $H^1(\Omega) \cap C_c(X)$ as core and for $u, v \in V$ we have

$$a_\mu(u, v) = \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\mu,$$

it is clear that $a_\mu(u, v) = 0$ whenever $u, v \in H^1(\Omega) \cap C_c(X)$ with $\operatorname{supp}[u] \cap \operatorname{supp}[v] = \emptyset$ and by Proposition 3.4.19 the form (a_μ, V) is local. $\square$

Now we can give the following result which characterizes the semigroups which are between $(e^{t\Delta_D})_{t\geq 0}$ and $(e^{t\Delta_N})_{t\geq 0}$.

Theorem 3.4.21. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $T = (T(t))_{t\geq 0}$ be a symmetric C_0-semigroup on $L^2(\Omega)$ satisfying*

$$e^{t\Delta_D} \leq T(t) \leq e^{t\Delta_N} \tag{3.15}$$

for all $t \geq 0$ in the sense of positive operators. Let $(a, D(a))$ be the closed form on $L^2(\Omega)$ associated with T. Then the following assertions are equivalent.

(i) $T(t) = e^{t\Delta_\mu}$ *for some positive Borel measure μ on $\partial\Omega$ which does not charge relatively polar Borel subsets of the part of $\partial\Omega$ on which it is locally finite.*

(ii) *(a)* $(a, D(a))$ *is local.*

(b) $D(a) \cap C_c(\bar{\Omega})$ *is dense in* $D(a)$.

To prove this result, we need the following version of Stone-Weierstrass' Theorem which is contained in [31, Theorem D.23].

Lemma 3.4.22 (Stone-Weierstrass). *Let X be a locally compact metric sapce, and let F be a subalgebra of $C_0(X)$ such that*

a) F separates the points of X, and *b) for all $x \in X$ there exists $u \in F$ such that $u(x) \neq 0$.*

Then F is uniformly dense in $C_0(X)$.

Proof of Theorem 3.4.21. (i) $\Rightarrow$ (ii). This part follows from Corollary 3.3.2 and Propositions 3.4.8 and 3.4.20.

(ii) $\Rightarrow$ (i). 1) Let

$$X := \bar{\Omega} \setminus \{x \in \bar{\Omega} : \ u(x) = 0 \ \forall \ u \in D(a) \cap C_c(\bar{\Omega})\}.$$

It is clear that X is relatively open. Since $H_0^1(\Omega) \subset D(a)$, we have $\Omega \subset X \subset \bar{\Omega}$. As $D(a)$ is a Dirichlet space, it follows from [55, Theorem 1.4.2 (ii)] that $D(a) \cap C_c(X)$ is a subalgebra of $C_c(X)$. It is also clear that $D(a) \cap C_c(X)$ separates the points of X. Moreover, by definition of X, we have that for every $x \in X$ there exists $u \in D(a) \cap C_c(X)$ such that $u(x) \neq 0$. It then follows from Lemma 3.4.22 that $D(a) \cap C_c(X)$ is uniformly dense in $C_c(X)$.

2) Since $T(t)$ satisfies (3.15), it follows from Theorem 3.4.2 that $H_0^1(\Omega)$ is an ideal of $D(a)$ and $D(a)$ is also an ideal of $\widetilde{H}^1(\Omega)$. Moreover, for all $u, v \in H_0^1(\Omega)$,

$$a(u, v) = \int_\Omega \nabla u \nabla v \, dx,$$

and for all $u, v \in D(a)_+$,

$$\int_\Omega \nabla u \nabla v \, dx := a_N(u, v) \leq a(u, v).$$

For $u, v \in D(a) \cap C_c(X)$ we let

$$b(u, v) := a(u, v) - a_N(u, v) = a(u, v) - \int_\Omega \nabla u \nabla v \, dx.$$

Let $\{G^a_\beta : \ \beta > 0\}$ be the resolvent of the operator associated with the closed form $(a, D(a))$ and $\{G^N_\beta : \ \beta > 0\}$ be the resolvent of Δ_N. Note that we regard G^a_β and G^N_β as operators on $L^2(X, m)$ (where m is defined in Section 2.2) which can be identified in an obvious way with $L^2(\Omega)$. With this consideration, the space X and the measure m satisfy the conditions (1.3).

Let $a^{(\beta)}$ and $a_N^{(\beta)}$ be the approximating forms of a and a_N as defined in (3.13) and let

$$\begin{aligned} b^{(\beta)}(u, v) &:= a^{(\beta)}(u, v) - a_N^{(\beta)}(u, v) \\ &= \beta(u - \beta G^a_\beta u, v) - \beta(u - \beta G^N_\beta u, v) \\ &= \beta(\beta(G^N_\beta - G^a_\beta)u, v). \end{aligned}$$

Since by the domination criterion, $b^{(\beta)}(u, v) \geq 0$ for all positive $u, v \in D(a) \cap C_c(X)$, we have that $\beta(G^N_\beta - G^a_\beta)$ is a positive symmetric operator on $L^2(X, m)$

and it then follows from Lemma 3.4.17 that there exists a positive Radon measure ν_β on $X \times X$ such that for $u, v \in D(a) \cap C_c(X)$ we have

$$b^\beta(u,v) = \beta(\beta(G_\beta^N - G_\beta^a)u, v) = \beta \int_{X\times X} u(x)v(y)\, d\nu_\beta.$$

It is clear that $b^\beta(u,v) \to b(u,v)$ as $\beta \uparrow \infty$ for all $u, v \in D(a) \cap C_c(X)$. Since for each $\beta > 0$ and $u \in D(a) \cap C_c(X)$

$$b^\beta(u,u) \le a(u,u),$$

it follows that the sequence of measures $(\beta\nu_\beta)$ is uniformly bounded on each compact subset of $X\times X$ and hence a subsequence $\beta_n\nu_{\beta_n}$ converges as $\beta_n \to \infty$ vaguely on $X \times X$ to a positive Radon measure ν. Moreover, since by 1) $D(a) \cap C_c(X)$ is uniformly dense in $C_c(X)$, we have that the measure ν is unique and therefore for all $u, v \in D(a) \cap C_c(X)$ we have

$$b(u,v) = \int_{X\times X} u(x)v(y)\, d\nu.$$

Since the forms a and a_N are local, it follows that $b(u,v) = 0$ for all $u, v \in D(a) \cap C_c(X)$ with $\operatorname{supp}[u] \cap \operatorname{supp}[v] = \emptyset$. This implies that $\operatorname{supp}[\nu] \subset \{(x,x) :\ x \in X\}$ and therefore

$$b(u,v) = \int_X u(x)v(x)\, d\nu.$$

Since $b(u,v) = 0$ for all $u, v \in \mathcal{D}(\Omega) \subset D(a)$, we have that $\operatorname{supp}[\nu] \subset X \setminus \Omega := \partial X$ and thus

$$b(u,v) = \int_{\partial X} u(x)v(x)\, d\nu.$$

Define a measure μ on $\partial\Omega$ by: for $B \in \mathcal{B}(\partial\Omega)$ we let

$$\mu(B) := \begin{cases} \nu(B) & \text{if } B \in \mathcal{B}(\partial X) = \mathcal{B}(\partial\Omega)_{\partial X}, \\ \infty & \text{if } B \notin \mathcal{B}(\partial X). \end{cases}$$

Then μ is a Borel measure on $\partial\Omega$ and for $u, v \in D(a) \cap C_c(X)$ we have

$$a(u,v) = \int_\Omega \nabla u \nabla v\, dx + \int_{\partial\Omega} uv\, d\mu. \tag{3.16}$$

Since $D(a)$ is an ideal of $\widetilde{H}^1(\Omega)$, it follows that $D(a) \cap C_c(X) = H^1(\Omega) \cap C_c(X)$ and therefore $a = a_\mu$ on $H^1(\Omega) \cap C_c(X)$. Since $(a, D(a))$ is a closed form, it follows that $(a, H^1(\Omega) \cap C_c(X)) = (a_\mu, H^1(\Omega) \cap C_c(X))$ is closable and by Corollary 3.3.2, this is equivalent to the property that $\mu|_{\partial X} = \nu$ does not charge relatively polar Borel subsets of ∂X.

3) Let (a_μ, V) be the closure of the form $(a_\mu, H^1(\Omega) \cap C_c(X))$. By definition, V is the completion of $H^1(\Omega) \cap C_c(X)$ with respect to the $a_\mu = a$-norm. To finish,

we have to show that $V = D(a)$. It is clear that V is a closed subspace of $D(a)$. It suffices to show that

$$D(a) \cap C_c(\bar{\Omega}) = \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : u|_{\partial\Omega \setminus \partial X} = 0, \int_{\partial X} |u|^2 \, d\mu < \infty\} =: E_\mu \tag{3.17}$$

and that (3.16) remains true for all $u, v \in E_\mu$.

In order to prove (3.17) it suffices to consider positive functions. Let $0 \le u \in E_\mu$. Then $(u-\varepsilon)^+ \in H^1(\Omega) \cap C_c(X)$ (by the fact that $D(a) \cap C_c(X) = H^1(\Omega) \cap C_c(X)$) for all $\varepsilon > 0$. Moreover, $(u-\varepsilon)^+ \to u$ in $H^1(\Omega)$ and $(u-\varepsilon)^+|_{\partial X} \to u|_{\partial X}$ in $L^2(\partial X, \mu)$ as $\varepsilon \downarrow 0$. Hence $(u-\varepsilon)^+$ is a Cauchy net in $D(a)$. Thus $u \in D(a)$ and

$$\begin{aligned} a(u,u) &= \lim_{\varepsilon \downarrow 0} a((u-\varepsilon)^+, (u-\varepsilon)^+) \\ &= \lim_{\varepsilon \downarrow 0} \left(\int_\Omega |\nabla (u-\varepsilon)^+|^2 \, dx + \int_{\partial X} ((u-\varepsilon)^+)^2 \, d\mu \right) \\ &= \int_\Omega |\nabla u|^2 \, dx + \int_{\partial X} |u|^2 \, d\mu. \end{aligned}$$

Conversely, let $0 \le u \in D(a) \cap C_c(\bar{\Omega})$. Since a is a Dirichlet form, $(u-\varepsilon)^+$ converges to u in $D(a)$ as $\varepsilon \downarrow 0$. Moreover, $(u-\varepsilon)^+ \in E_\mu$. Hence

$$a(u,u) = \lim_{\varepsilon \downarrow 0} a((u-\varepsilon)^+, (u-\varepsilon)^+) = \int_\Omega |\nabla u|^2 \, dx + \int_{\partial X} |u|^2 \, d\mu.$$

We have proved (3.17) and (3.16) for $u = v$. The polarization identity shows that (3.16) holds for all $u, v \in E_\mu$. Now Proposition 3.4.8 implies that $a = a_\mu$ and therefore $V = D(a)$. □

Next we characterize those semigroups which are given by finite measures.

Corollary 3.4.23. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $T = (T(t))_{t \ge 0}$ be a symmetric C_0-semigroup on $L^2(\Omega)$ satisfying*

$$e^{t\Delta_D} \le T(t) \le e^{t\Delta_N}$$

for all $t \ge 0$ in the sense of positive operators. Let $(a, D(a))$ be the closed form on $L^2(\Omega)$ associated with T. Then the following assertions are equivalent.

(i) $T(t) = e^{t\Delta_\mu}$ for a unique finite Borel measure μ on $\partial\Omega$ in $\mathcal{M}_0$.

(ii) (a) $(a, D(a))$ is local.
(b) $1 \in D(a) \cap C(\bar{\Omega})$.

Proof. (i) $\Rightarrow$ (ii). This part follows from Theorem 3.3.1, Corollary 3.4.6 and Proposition 3.4.20.

(ii) $\Rightarrow$ (i). The proof is similar to the proof of Theorem 3.4.21.

1) Since $1 \in D(a)$ and $D(a)$ is an ideal of $\widetilde{H}^1(\Omega)$, it follows that $D(a) \cap C(\bar{\Omega}) = H^1(\Omega) \cap C(\bar{\Omega})$ which is uniformly dense in $C(\bar{\Omega})$.

2) The fact that $1 \in D(a) \cap C(\bar{\Omega})$ implies that $X = \bar{\Omega}$ which is a compact set. Proceeding as in the proof of Theorem 3.4.21 by letting

$$b(u,v) = a(u,v) - \int_{\Omega} \nabla u \nabla v \, dx$$

for $u, v \in H^1(\Omega) \cap C(\bar{\Omega})$, we obtain that there exists a unique positive Radon measure μ on $\partial\Omega$; i.e., a finite measure such that for all $u, v \in H^1(\Omega) \cap C(\bar{\Omega})$ we have

$$b(u,v) = \int_{\partial\Omega} uv \, d\mu$$

and therefore for all $u, v \in H^1(\Omega) \cap C(\bar{\Omega})$ we have

$$a(u,v) = \int_{\Omega} \nabla u \nabla v \, dx + \int_{\partial\Omega} uv \, d\mu.$$

Since the form $(a, D(a))$ is closed, it follows that the form $(a, H^1(\Omega) \cap C(\bar{\Omega})) = (a_\mu, H^1(\Omega) \cap C(\bar{\Omega}))$ is closable on $L^2(\Omega)$ and by Theorem 3.3.1 this is equivalent to the property that $\mu \in \mathcal{M}_0$.

3) Let (a_μ, V) be the closure of $(a_\mu, H^1(\Omega) \cap C(\bar{\Omega}))$. To finish, we have to show that $(a, D(a)) = (a_\mu, V)$. Since $\mu \in \mathcal{M}_0$, it follows from Proposition 3.3.4 that

$$V = \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} \in L^2(\partial\Omega, \mu)\}.$$

It is clear that V is a closed subspace of $D(a)$. Let $u \in D(a)$. Without restriction, we assume that u is r.q.c. By considering u^+ and u^- separately if necessary, we may assume that $0 \le u$. For $k \in \mathbb{N}$ we let $u_k := u \wedge k$. Then $u_k \in \widetilde{H}^1(\Omega)$ is r.q.c. Since $0 \le u_k \le k$ and $\mu(\partial\Omega) < \infty$, it follows that $u_k \in L^2(\partial\Omega, \mu)$ and therefore $u_k \in V$. It is also clear that $u_k \to u$ in $\widetilde{H}^1(\Omega)$ and thus after taking a subsequence if necessary, we may assume that $u_k \to u$ r.q.e on $\bar{\Omega}$. Since $\mu \in \mathcal{M}_0$, it follows that $u_k \to u$ μ a.e. on $\partial\Omega$. Finally, since $0 \le u_k \le k$, the Lebesgue Dominated Convergence Theorem implies that $u_k \to u$ in $L^2(\partial\Omega, \mu)$ and thus $u_k \to u$ with respect to the a_μ-norm and therefore $u \in V$ which completes the proof. $\square$

3.5 Convergence of Forms.

Throughout this section, Ω will denote a bounded open set in $\mathbb{R}^N$. Let $\mu_n \in \mathcal{M}_0$ be a sequence of measures on $\partial\Omega$ and $\mu \in \mathcal{M}_0$ be a measure on $\partial\Omega$. Consider the submarkovian C_0-semigroups $(e^{t\Delta_{\mu_n}})_{t \ge 0}$ and $(e^{t\Delta_\mu})_{t \ge 0}$ on $L^2(\Omega)$ generated respectively by Δ_{μ_n} and Δ_μ. The basic question which we address is, whether we have

$$e^{t\Delta_{\mu_n}} \to e^{t\Delta_\mu} \quad \text{as} \quad n \to \infty \tag{3.18}$$

if the measures μ_n converge to μ in an appropriate sense. By the Trotter-Kato Approximation Theorem (see [11, Theorem 3.6.1 p.149]), we have strong convergence in (3.18) if the generators $\Delta_{\mu_n} \to \Delta_\mu$ as $n \to \infty$ in the strong resolvent sense. Before studying this notion, we recall the following well-known result [84, Theorem S.17 p.374].

Proposition 3.5.1. *Let A and B be two selfadjoint closed operators on a Hilbert space H and let a and b be the corresponding forms. Then $a \le b$ if and only if $(B+1)^{-1} \le (A+1)^{-1}$.*

We begin with a very simple case by considering $\mu \in \mathcal{M}_0$ to be a finite Borel measure on $\partial\Omega$. Before, we need the following notions.

Definition 3.5.2. *Let μ be a Borel measure on $\partial\Omega$ which is in $\mathcal{M}_0$. A set F is said to be a* **relative quasi-support of** μ *if the following two conditions are satisfied:*

(i) F is relatively quasi-closed and $\mu(\partial\Omega \setminus F) = 0$.

(ii) If $\widetilde{F}$ is another set with property (i) then $F \subset \widetilde{F}$ r.q.e.

The relative quasi-support F of μ is unique up to the r.q.e. equivalence. Moreover if Γ is the **topological support** of μ, since every relatively closed set is relatively quasi-closed, we have that $F \subset \Gamma$ r.q.e. and by deleting a set of zero relative capacity from F if necessary, we can always assume that $F \subset \Gamma$.

Thoughout the following for a measure μ on $\partial\Omega$ in $\mathcal{M}_0$ we always denote by (a_μ, V) the closed form on $L^2(\Omega)$ as defined in the preceding section; i.e.

$$a_\mu(u,v) := \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\mu$$

with domain

$$V := \{u \in \widetilde{H}^1(\Omega) : \int_{\partial\Omega} |\tilde{u}| \, d\mu < \infty\}.$$

Proposition 3.5.3. *Let $\mu \in \mathcal{M}_0$ be a finite Borel measure on $\partial\Omega$ with relative quasi-support $\partial\Omega$ and let $k \in \mathbb{N}$. We define*

$$a_{\mu k}(u,v) := \int_\Omega \nabla u \nabla v \, dx + k \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\mu \quad u, v \in V.$$

Then $\Delta_{\mu k} \to \Delta_D$ as $k \to \infty$ in the strong resolvent sense.

Proof. It is clear that $0 \le a_{\mu 1} \le a_{\mu 2} \le \ldots \le a_{\mu k} \le \ldots$. Let

$$V_\infty := \{u \in V : \sup_k a_{\mu k}(u,u) < \infty\}$$

and

$$a_\infty(u,u) := \lim_{k \to \infty} a_{\mu k}(u,u) = \sup_k a_{\mu k}(u,u).$$

1) First, we prove that $(a_\infty, V_\infty) = (a_D, H_0^1(\Omega))$. It is easy to see that $H_0^1(\Omega) \subset V_\infty$. Let us prove the converse inclusion. Let $u \in V_\infty$. Then $\lim_{k\to\infty} a_{\mu k}(u,u) < \infty$. Without restriction, we may assume that u is r.q.c. This implies that

$$\lim_{k\to\infty} k \int_{\partial\Omega} |u|^2 \, d\mu < \infty,$$

which is possible if and only if $\int_{\partial\Omega} |u|^2 \, d\mu = 0$ and then $u = 0$ μ a.e. on $\partial\Omega$. Since we suppose that the relative quasi-support of μ is $\partial\Omega$, this implies that $u|_{\partial\Omega} = 0$ r.q.e. (this follows from an abstract result contained in [55, Theorem 4.6.2]) which implies that $u \in H_0^1(\Omega)$ by Theorem 2.4.1.

2) Now we show that $\Delta_{\mu k} \to \Delta_D$ as $k \to \infty$ in the strong resolvent sense. Since $a_{\mu k} \le a_\infty$, by Proposition 3.5.1, for $\varphi \in L^2(\Omega)$, we have

$$(\varphi, (\Delta_D + 1)^{-1}\varphi) \le (\varphi, (\Delta_{\mu k} + 1)^{-1}\varphi).$$

Since $(\varphi, (\Delta_{\mu k} + 1)^{-1}\varphi)$ is monotone decreasing, it follows that

$$\lim_{k\to\infty} (\varphi, (\Delta_{\mu k} + 1)^{-1}\varphi) = \inf_k (\varphi, (\Delta_{\mu k} + 1)^{-1}\varphi)$$

has a nonzero value, so we can find a selfadjoint operator C with zero kernel so that $(\varphi, (\Delta_{\mu k}+1)^{-1}\varphi) \to C$ weakly as $k \to \infty$ (see the proof of [84, Theorem S.14 p.376]). Let b be the form associated with $B := C^{-1} - 1$. Since

$$(\Delta_D + 1)^{-1} \le C \le (\Delta_{\mu k} + 1)^{-1},$$

by Proposition 3.5.1, we have that $a_{\mu k} \le b \le a_D$. Passing to the limit as $k \to \infty$, we obtain $a_\infty = a_D \le b \le a_D$. Then $b = a_D$ and $B = \Delta_D$. Thus $(\Delta_{\mu k} + 1)^{-1} \to (\Delta_D + 1)^{-1}$ weakly as $k \to \infty$. By a similar argument this holds if 1 is replaced by an arbitrary $\lambda > 0$ and by analyticity we have weak convergence of the resolvent on $\mathbb{C} \setminus [0,\infty)$ and since weak resolvent convergence implies strong resolvent convergence (see [84, Section VIII.7]), this proves the claim. □

Proposition 3.5.4. *Let $\mu \in \mathcal{M}_0$ be a finite Borel measure on $\partial\Omega$, $k \in \mathbb{N}^*$ and define*

$$a_{\mu k}(u,v) := \int_\Omega \nabla u \nabla v \, dx + \frac{1}{k} \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\mu \quad u, v \in V.$$

Then $\Delta_{\mu k} \to \Delta_N$ as $k \to \infty$ in the strong resolvent sense.

We do not give a proof of Proposition 3.5.4. We shall consider the case where the measure on $\partial\Omega$ is the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure which we denote by σ. Then Proposition 3.5.4 becomes a particular case, since σ is not always a Radon measure. For this we need some preparations.

Recall that, we call Δ_N (the Laplacian with Neumann boundary conditions), the selfadjoint operator on $L^2(\Omega)$ associated with the closed form a_N with domain $\widetilde{H}^1(\Omega)$ defined by

$$a_N(u,v) := \int_\Omega \nabla u \nabla v \, dx.$$

If we replace $\widetilde{H}^1(\Omega)$ by the closed subspace D_B of the form:

$$D_B := \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \text{ q.e. on } \ B\}$$

for some $B \in \mathcal{B}(\partial\Omega)$ with $\mathrm{Cap}_{\bar{\Omega}}(\partial\Omega \backslash B) > 0$, then the operator Δ_N^B associated with (a_N, D_B) is the Laplacian with Dirichlet boundary conditions on B and Neumann boundary conditions on $\partial\Omega \setminus B$ which we call **Dirichlet-Neumann Laplacian**.

In the following, without restriction, we assume that Ω is such that $\sigma \in \mathcal{M}_0$.

Proposition 3.5.5. *Let $k \in \mathbb{N}^*$ and define*

$$a_{\sigma k}(u, v) := \int_\Omega \nabla u \nabla v \, dx + \frac{1}{k} \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\sigma, \qquad u, v \in V.$$

Let $\Delta_{\sigma k}$ be the operator associated with $(a_{\sigma k}, V)$. If $\sigma(\partial\Omega) < \infty$, then $\Delta_{\sigma k} \to \Delta_N$ as $k \to \infty$ in the strong resolvent sense. If σ is locally infinite on a part $\Gamma_\infty \subset \partial\Omega$, then $\Delta_{\sigma k} \to \Delta_N^{\Gamma_\infty}$ as $k \to \infty$ in the strong resolvent sense.

Proof. We give a proof only for the case $\sigma(\partial\Omega) < \infty$. The proof of the other case is exactly the same.

a) We have $a_{\sigma 1} \geq a_{\sigma 2} \geq \ldots \geq a_{\sigma k} \geq \ldots \geq 0$. For $u \in V$ define

$$a_\infty(u, u) := \lim_{k \to \infty} a_{\sigma k}(u, u) = \inf_k a_{\sigma k}(u, u).$$

It is clear that for such u,

$$a_\infty(u, u) = \int_\Omega |\nabla u|^2 \, dx.$$

Let $((a_\infty)_r, V)$ be the closable part of (a_∞, V). Since (a_∞, V) is closable on $L^2(\Omega)$, we have $(a_\infty)_r = a_\infty$. Let V_∞ be the completion of V with respect to the norm $\|u\|_{a_\infty} = \|u\|_{H^1(\Omega)}$. By Corollary 3.4.6, $V_\infty = \widetilde{H}^1(\Omega)$. Thus the closed form $(\overline{(a_\infty)_r}, V_\infty)$ is the form associated with the operator Δ_N.

b) Let us prove that $\Delta_{\sigma k} \to \Delta_N$ as $k \to \infty$ in the strong resolvent sense. Let $\varphi \in L^2(\Omega)$. Since $(\varphi, (\Delta_{\sigma k}+1)^{-1}\varphi)$ is monotone increasing and $(\varphi, (\Delta_{\sigma k}+1)^{-1}\varphi) \leq (\varphi, (\Delta_N + 1)^{-1}\varphi)$, it follows as in the proof of Proposition 3.5.3 that there exists a closed operator C with zero kernel such that

$$(\Delta_{\sigma k} + 1)^{-1} \to C \quad \text{weakly.}$$

Let s be the quadratic form of $S := C^{-1} - 1$. Then $s \leq a_{\sigma k}$. Since s is a closed form, it follows that $s = s_r$. This implies that $s \leq (a_\infty)_r$ and thus $s \leq \overline{(a_\infty)_r}$. On the other hand, since $\overline{(a_\infty)_r} \leq a_\infty \leq a_{\sigma k}$, we have that $(\Delta_N + 1)^{-1} \geq (\Delta_{\sigma k} + 1)^{-1}$. Taking the limit as $k \to \infty$, one obtains $(\Delta_N + 1)^{-1} \geq (S + 1)^{-1}$ so $s \geq \overline{(a_\infty)_r}$; i.e., $s = \overline{(a_\infty)_r} = a_N$. Now the statement follows from the fact that weak resolvent convergence implies strong resolvent convergence. $\square$

Next we consider the case of a sequence of measures.

Proposition 3.5.6. *Let μ_n and μ be finite Borel measures on $\partial\Omega$ in $\mathcal{M}_0$. Assume that μ_n is monotone and $\mu_n \to \mu$ vaguely. Then $\Delta_{\mu_n} \to \Delta_\mu$ in the strong resolvent sense.*

Proof. We give a proof only for the increasing case.

1) We have again $0 \leq a_{\mu_1} \leq a_{\mu_2} \leq \ldots \leq a_{\mu_n} \leq \ldots$. We will denote by V_n the domain of the form a_{μ_n}. Let

$$V_\infty := \{u \in \cap_n V_n : \sup_n a_{\mu_n}(u,u) < \infty\}$$

and $a_\infty(u,u) := \lim_{n\to\infty} a_{\mu_n}(u,u)$. It is clear that

$$\ldots \subset V_{n+1} \subset V_n \subset \ldots \subset V_2 \subset V_1.$$

We want to prove that $a_\infty = a_\mu$. As $H^1(\Omega) \cap C(\bar{\Omega}) \subset V_n$ for all $n \geq 1$ we have that $H^1(\Omega) \cap C(\bar{\Omega}) \subset V_\infty$. Let $u \in H^1(\Omega) \cap C(\bar{\Omega})$. Since $\mu_n \to \mu$ vaguely, it follows that $\lim_{n\to\infty} a_{\mu_n}(u,u) = a_\mu(u,u)$. Since $H^1(\Omega) \cap C(\bar{\Omega})$ is dense in V_∞, we obtain that

$$\lim_{n\to\infty} a_{\mu_n}(u,u) := a_\infty(u,u) = a_\mu(u,u) \quad \forall\, u \in V_\infty.$$

Then $a_\mu(u,u) = a_\infty(u,u)$ for all $u \in V_\infty$. Since $a_\mu \leq a_\infty$ we have that $V_\infty \subset V_\mu$. Let $u \in V_\mu$. It is clear that $u \in \cap_n V_n$. The density of $H^1(\Omega) \cap C(\bar{\Omega})$ in V_μ implies that $\sup_n a_{\mu_n}(u,u) = a_\mu(u,u) < \infty$ and thus $u \in V_\infty$ which implies that $V_\infty = V_\mu$.

2) The proof of $\Delta_{\mu_n} \to \Delta_\mu$ in the strong resolvent sense is the same as in Proposition 3.5.3. □

Next we introduce the following well-known notion of convergence.

Definition 3.5.7. *Let X be a metric space. Let (F_n) be a sequence of functions from $X \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $F : X \to \bar{\mathbb{R}}$. We say that (F_n) Γ-**converges** to F in X if the following conditions are satisfied.*

a) For every $u \in X$ and for every sequence u_n converging to u in X

$$F(u) \leq \liminf_{n\to\infty} F_n(u_n).$$

b) For every $u \in X$ there exists a sequence u_n converging to u in X such that

$$F(u) = \limsup_{n\to\infty} F_n(u_n).$$

For each $\mu \in \mathcal{M}_0$ we associate the following functional F_μ defined on $L^2(\Omega)$ by letting

$$F_\mu(u) := \begin{cases} \int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} |\tilde{u}|^2\, d\mu & u \in V \\ \infty & u \in L^2(\Omega) \text{ but not in } V. \end{cases}$$

We only consider measures which do not charge relatively polar sets.

Definition 3.5.8. *Let $\mu_n, \mu \in \mathcal{M}_0$. We say that μ_n γ-**converges** to μ if the sequence of functionals F_{μ_n} Γ-converges to the functional F_μ in $L^2(\Omega)$.*

Proposition 3.5.9. *Let $\mu_n, \mu \in \mathcal{M}_0$. Assume that μ_n γ-converges to μ. Then $\Delta_{\mu_n} \to \Delta_\mu$ in the strong resolvent sense.*

Proof. Let $f \in L^2(\Omega),\ u_n = \lambda R(\lambda, \Delta_{\mu_n})f$ and $u = \lambda R(\lambda, \Delta_\mu)f$ where $\lambda > 0$. We have to prove that $u_n \to u$ strongly in $L^2(\Omega)$ as $n \to \infty$. Remark that u_n is a weak solution of the equation $-\Delta_{\mu_n} u_n + \lambda u_n = \lambda f$. The **Dirichlet principle** (see [24, Proposition IX.22]) says that u_n is given by

$$\min_{u \in \widetilde{H}^1(\Omega)} \left\{ \frac{1}{2} \left(\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} |\tilde{u}|^2 \, d\mu + \lambda \int_\Omega |u|^2 \, dx \right) - \lambda \int_\Omega fu \, dx \right\}.$$

By the definition of Γ-convergence there exists a sequence v_n converging to u in $L^2(\Omega)$ and

$$F_\mu(u) = \lim_{n \to \infty} F_{\mu_n}(v_n).$$

Set

$$J_\mu(u) := F_\mu(u) + \lambda \int_\Omega |u|^2 \, dx - 2\lambda \int_\Omega fu \, dx.$$

Since u_n is minimum, we have

$$J_{\mu_n}(u_n) \le J_{\mu_n}(v_n).$$

Therefore

$$J_{\mu_n}(u_n) + \lambda \int_\Omega |f|^2 \, dx \le J_{\mu_n}(v_n) + \lambda \int_\Omega |f|^2 \, dx.$$

This means that

$$F_{\mu_n}(u_n) + \lambda \int_\Omega (u_n - f)^2 \, dx \le F_{\mu_n}(v_n) + \lambda \int_\Omega (v_n - f)^2 \, dx.$$

Hence,

$$\begin{aligned} \limsup_{n \to \infty} \left(F_{\mu_n}(u_n) + \lambda \int_\Omega (u_n - f)^2 \, dx \right) &\le \lim_n \left(F_{\mu_n}(v_n) + \lambda \int_\Omega (v_n - f)^2 \, dx \right) \\ &= F_\mu(u) + \lambda \int_\Omega (u - f)^2 \, dx. \end{aligned}$$

It is clear that u_n is a bounded sequence in V. Then there exists a subsequence which converges weakly in V to a function $w \in V$. We may assume that $u_n \to w$ weakly in V. Then $F_\mu(w) \le \liminf_{n \to \infty} F_{\mu_n}(u_n)$. Moreover,

$$\int_\Omega (w - f)^2 \, dx \le \liminf_{n \to \infty} \int_\Omega (u_n - f)^2 \, dx.$$

Hence,

$$\begin{aligned} F_\mu(w) + \lambda \int_\Omega (w-f)^2\,dx &\le \liminf_{n\to\infty} \left(F_{\mu_n}(u_n) + \lambda \int_\Omega (u_n - f)^2\,dx \right) \\ &\le \limsup_{n\to\infty} \left(F_{\mu_n}(u_n) + \lambda \int_\Omega (u_n - f)^2\,dx \right) \\ &\le F_\mu(u) + \lambda \int_\Omega (u-f)^2\,dx. \end{aligned}$$

Since u is unique, we have that $w = u$ and

$$F_\mu(u) + \lambda \int_\Omega (u-f)^2\,dx = \lim_{n\to\infty} \left(F_{\mu_n}(u_n) + \lambda \int_\Omega (u_n - f)^2\,dx \right).$$

This implies that $\int_\Omega (u-f)^2\,dx = \lim_{n\to\infty} \int_\Omega (u_n - f)^2\,dx$ and thus $u_n \to u$ strongly in $L^2(\Omega)$ as $n \to \infty$. □

3.6 Comments.

Sections 3.1 and 3.2.
A proof of Lemma 3.2.5 is given by Daners in [34] for the particular case where μ is the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure σ.

Now we prove the following result which says that the space $H^1(\Omega)$ is a lattice and the lattice operations are continuous. The first part of the proof is contained in [59, Lemma 7.6 p.152] and the second part in [33, Lemma 6.4.1].

Theorem 3.6.1. *The following statements are satisfied.*

a) If $u \in H^1(\Omega)$ then $u^+ \in H^1(\Omega)$ and the mapping $u \mapsto |u|$ is continuous from $H^1(\Omega)$ to $H^1(\Omega)$.

b) If $0 \le u \in H^1(\Omega)$ then $(u-1)^+ \in H^1(\Omega)$ and the mapping $u \mapsto (u-1)^+$ is continuous from $H^1(\Omega)_+$ into $H^1(\Omega)_+$.

Proof. a) Let $u \in H^1(\Omega)$. It is clear that $u^+ \in L^2(\Omega)$. For every $\varepsilon > 0$ let

$$f_\varepsilon(\xi) := \begin{cases} (\xi^2 + \varepsilon^2)^{1/2} - \varepsilon & \text{if } \xi > 0 \\ 0 & \text{if } \xi \le 0. \end{cases}$$

For every $\varepsilon > 0$, $f_\varepsilon \in C^1(\mathbb{R})$ and f'_ε is bounded since

$$f'_\varepsilon(\xi) = \xi(\xi^2 + \varepsilon^2)^{-1/2}, \ \ \xi > 0.$$

Thus $f_\varepsilon \circ u \in H^1(\Omega)$ for all $u \in H^1(\Omega)$. For $\varphi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega (f_\varepsilon \circ u) D_i\varphi\,dx = -\int_{u>0} \varphi \frac{u D_i u}{(u^2 + \varepsilon^2)^{1/2}}\,dx.$$

Taking the limit as $\varepsilon \to 0$ we obtain

$$\int_\Omega u^+ D_i\varphi\, dx = -\int_{u>0} \varphi D_i u\, dx$$

which implies that

$$D_i u^+ = \begin{cases} D_i u & \text{if } u > 0 \\ 0 & \text{fi } u \le 0 \end{cases} \tag{3.19}$$

and thus $D_i u^+ \in L^2(\Omega)$. Notice that by (3.19) we have $\nabla|u| = \operatorname{sgn} u \nabla u$. Hence $\| \, |u| \, \|_{H^1(\Omega)} = \|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$. To prove the continuity of the mapping, let (u_n) be a sequence in $H^1(\Omega)$ converging to v in $H^1(\Omega)$. Then, by the reverse triangle inequality

$$| \, \| \, |u_n| \, \| - \| \, |v| \, \| \, | = | \, \|u_n\| \| - \|v\| \, | \le \|u_n - v\|_{H^1(\Omega)}.$$

This implies that $\| \, |u_n| \, \|_{H^1(\Omega)}$ converges to $\| \, |v| \, \|_{H^1(\Omega)}$ as $n \to \infty$. On the other hand, as $\| \, |u_n| \, \|_{H^1(\Omega)} = \|u_n\|_{H^1(\Omega)}$ the sequence $(|u_n|)$ is bounded in $H^1(\Omega)$, and therefore has a weakly convergent subsequence. As the mapping $u \mapsto |u|$ is continuous in $L^2(\Omega)$ its (unique) limit is $|v|$ in $L^2(\Omega)$, and therefore the whole sequence converges weakly to $|v|$ in $H^1(\Omega)$. As we showed already that the norm converges, we conclude that $|u_n| \to |v|$ in $H^1(\Omega)$, proving the continuity of the mapping $u \mapsto |u|$.

b) The proof is similar as in a). Here we let

$$f_\varepsilon(\xi) = \begin{cases} ((\xi - 1)^2 + \varepsilon^2)^{1/2} - \varepsilon & \text{if } \xi > 1 \\ 0 & \text{if } \xi \le 1 \end{cases}$$

and use **Stampacchia's Lemma** ($\nabla u = 0$ a.e. on $\{x : \ u(x) = c\}$ for any constant c). □

Section 3.3.

The notion of perturbation of regular Dirichlet forms by measures has been considered in [55, Section 6.1] and [89].

Lemma 3.3.5 can be also proved by noticing that the set $\mathcal{R} \cap \mathcal{M}_0$ is an ideal in $\mathcal{R}$ where $\mathcal{R}$ denotes the set of all Radon measures on $\partial\Omega$. In fact, let $\mu \in \mathcal{R} \cap \mathcal{M}_0$ and $\nu \in \mathcal{R}$ be such that $\nu \le \mu$. Let $A \in \mathcal{B}(\partial\Omega)$ be such that $\operatorname{Cap}_{\bar{\Omega}}(A) = 0$. Since $\mu \in \mathcal{R} \cap \mathcal{M}_0$, we have that $\mu(A) = 0$. But $\nu \le \mu$ implies $\nu(A) = 0$ and thus $\nu \in \mathcal{R} \cap \mathcal{M}_0$. Using the **Riesz Decomposition Theorem** (see [86, Theorem 2.10 p.62]), the band orthogonal to $\mathcal{R} \cap \mathcal{M}_0$ (in the lattice sense) is the set of all such measures μ_s. This technique has been used by Stollmann and Voigt in [89].

Section 3.4.

By Proposition 3.4.8, for a given Borel measure μ on $\partial\Omega$ in $\mathcal{M}_0$, there always exists a relatively open set X satisfying $\Omega \subset X \subset \bar{\Omega}$ such that (a_μ, V) is regular on X. Note that V is an ideal of $\widetilde{H}^1(\Omega)$ containing $H^1_0(\Omega)$ as a closed subideal. A natural question is the following. Let I be an ideal of $\widetilde{H}^1(\Omega)$ containing $H^1_0(\Omega)$ as a closed subideal. Is the space I regular on some subset Y of $\bar{\Omega}$? By [88, Theorem 1.1] if I is a closed subspace of $\widetilde{H}^1(\Omega)$ then there exists a Borel subset M of $\partial\Omega$ such that

$$I = \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \text{ r.q.e. on } \ M\}.$$

A closer look of the proof of [88, Theorem 1.1] shows that M is in fact relatively quasi-closed and therefore $\bar{\Omega} \setminus M$ is relatively quasi-open. If $\bar{\Omega} \setminus M$ is relatively open (i.e., if M is relatively closed) by Theorem 2.4.2 I is regular on $X := \bar{\Omega} \setminus M$. Since a relatively quasi-open set is not necessarily relatively open, we do not know whether I is in general regular. Therefore we do not know if condition (ii) (b) of Theorem 3.4.21 can be omitted. By Example 3.4.15, the local condition on the form can not be omitted.

Finally, the notion of local forms is contained in [55] and [69]. It is well-known that if $(a, D(a))$ is a regular local symmetric Dirichlet form on $L^2(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$ and if $\mathcal{D}(\Omega) \subset D(a)$ then the restriction of a to $\mathcal{D}(\Omega)$ is of the form

$$a(u, v) = \sum_{i,j=1}^{N} \int_\Omega D_i u D_j v \, d\nu_{ij} + \int_\Omega uv \, dk$$

where for $1 \le i, j \le N$, ν_{ij} is a Radon mesure on Ω such that for every compact set $K \subset \Omega$, $\nu_{ij}(K) = \nu_{ji}(K)$ and $\sum_{i,j=1}^N \xi_i \xi_j \nu_{ij}(K) \ge 0$ for all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ and here k is a positive Radon measure on Ω (see [69, II Theorem 2.8 p.47]).

Section 3.5.

The monotone convergence of forms is contained in [84]. For more information on the notion of Γ-convergence we refer to [21] or [32].

Chapter 4

Robin and Neumann Boundary Conditions

In this chapter, we examine two particular cases of Chapter 3. The first one, Robin boundary conditions, corresponds to the case where μ is the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$ which we denote by σ, or more generally it corresponds to the case where μ is absolutely continuous with respect to σ. The second one, Neumann boundary conditions, corresponds to the cases where $\mu = 0$ or μ is concentrated on a relatively polar Borel subset of $\partial\Omega$. We need some preparations to examine these cases.

4.1 Maz'ya Inequality.

Before giving the very remarkable inequality due to Maz'ya (Theorem 4.1.7) and called Maz'ya inequality, we need some preparations. The proof of the Maz'ya inequality is based on Theorem 4.1.1, and on the well-known coarea formula (Theorem 4.1.4) and on the isoperimetric inequality. The proof of Theorem 4.1.1 given here is taken from [43, Remark p.192].

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in C^\infty(\Omega)$. Let*

$$E_t := \{x \in \Omega : u(x) \geq t\},\ t \in \mathbb{R}.$$

Then

$$\|u\|_{\frac{N}{N-1}} \leq \int_0^\infty |E_t|^{\frac{N-1}{N}}\ dt. \tag{4.1}$$

Proof. We can assume that $u \geq 0$. Let

$$u_t := \min(t, u) \ \text{ and } \ F(t) := \left(\int_\Omega u_t^{\frac{N}{N-1}}\ dx\right)^{\frac{N-1}{N}}.$$

Then F is nondecreasing on $(0,\infty)$ and

$$\lim_{t\to\infty} F(t) = \|u\|_{\frac{N}{N-1}}.$$

Moreover, for $h > 0$,

$$\begin{aligned} 0 \le F(t+h) - F(t) &= \left[\int_\Omega u_{t+h}^{\frac{N}{N-1}}\,dx\right]^{\frac{N-1}{N}} - \left[\int_\Omega u_t^{\frac{N}{N-1}}\,dx\right]^{\frac{N-1}{N}} \\ &\le \left[\int_\Omega |u_{t+h} - u_t|^{\frac{N}{N-1}}\,dx\right]^{\frac{N-1}{N}} \\ &\le h|E_t|^{\frac{N-1}{N}}. \end{aligned}$$

Thus $\frac{F(t+h)-F(t)}{h} \le |E_t|^{\frac{N-1}{N}}$. Taking the limit as $h \to 0$ (since F is locally Lipschitz it is differentiable a.e.) we obtain that $F'(t) \le |E_t|^{\frac{N-1}{N}}$. Integrating, we obtain

$$\left(\int_\Omega u^{\frac{N}{N-1}}\,dx\right)^{\frac{N-1}{N}} = \int_0^\infty F'(t)\,dt \le \int_0^\infty |E_t|^{\frac{N-1}{N}}\,dt$$

which gives (4.1). $\square$

Lemma 4.1.2. *Let $(X, \mathcal{B}, \mu)$ be a measure space and $u : X \to \mathbb{R}$ a nonnegative function. Then*

$$\int_X u(x)\,d\mu = \int_0^\infty \mu(E_t)\,dt$$

where the integral over $[0,\infty)$ is an improper Riemann integral and $E_t := \{x \in X :\ u(x) \ge t\}$.

Proof. Let $A := \{(x,t) : u(x) \ge t\}$ and $\mathcal{L}^1$ be the Lebesgue measure on $\mathbb{R}$. Then

$$\begin{aligned} \int_0^\infty \mu(E_t)\,dt &= \int_0^\infty \mu(\{x : (x,t) \in A\})\,dt \\ &= \int_X \mathcal{L}^1(\{t \in [0,\infty) : (x,t) \in A\})\,d\mu \\ &= \int_X \mathcal{L}^1([0,u(x)])\,d\mu \\ &= \int_X u(x)\,d\mu. \end{aligned}$$

$\square$

A proof of the following result due to Morse is contained in [73, Corollary 1.2.2].

Lemma 4.1.3 (Morse). *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in C^\infty(\Omega)$. Then for almost all $t \in \mathbb{R}$ the set $\{x \in \Omega : \ u(x) = t\}$ is a smooth manifold.*

The following well-known result called Coarea Formula is contained in [43] or [50]. The proof given here is taken from [73, Theorem 1.2.4].

Theorem 4.1.4 (Coarea formula). *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\varphi \in C(\Omega)$, $\varphi \geq 0$. Let $u \in C^\infty(\Omega)$. Then*

$$\int_\Omega \varphi(x)|\nabla u(x)|\,dx = \int_0^\infty \int_{A_t} \varphi(x)\,d\mathcal{H}^{N-1}\,dt \tag{4.2}$$

where $A_t := \{x \in \Omega : |u(x)| = t\}$. Here we may assume $\mathcal{H}^{N-1}$ to be the usual Lebesgue surface measure, since by Lemma 4.1.3, A_t is a smooth manifold.

Proof. Let $\omega \in \mathcal{D}(\Omega)^N := \mathcal{D}(\Omega, \mathbb{R}^N)$ and $u \in C^\infty(\Omega)$. Integrating by parts yields,

$$\int_\Omega \omega \nabla u\,dx = -\int_\Omega u \operatorname{div}\omega\,dx.$$

Since for $u \geq 0$, $u(x) = \int_0^\infty \chi_{[u\geq t]}(x)\,dt$, we obtain

$$\begin{aligned}
\int_\Omega u \operatorname{div}\omega\,dx &= \int_\Omega \left(\int_0^\infty \chi_{[u\geq t]}(x)\,dt\right) \operatorname{div}\omega(x)\,dx \\
&= \int_0^\infty \left(\int_\Omega \chi_{[u\geq t]}(x) \operatorname{div}\omega(x)\,dx\right)\,dt \\
&= \int_0^\infty dt \int_{u\geq t} \operatorname{div}\omega(x)\,dx.
\end{aligned}$$

For $u \leq 0$, we have $u(x) = \int_{-\infty}^0 (\chi_{[u\geq t]}(x) - 1)\,dt$. Then

$$\begin{aligned}
\int_\Omega u \operatorname{div}\omega\,dx &= \int_\Omega \left(\int_{-\infty}^0 \big(\chi_{[u\geq t]}(x) - 1\big)\,dt\right) \operatorname{div}\omega(x)\,dx \\
&= \int_{-\infty}^0 dt \int_{u\geq t} \operatorname{div}\omega(x)\,dx \\
&= -\int_{-\infty}^0 dt \int_{u\leq t} \operatorname{div}\omega(x)\,dx.
\end{aligned}$$

Finally we obtain that

$$\begin{aligned}
\int_\Omega \omega \nabla u\,dx &= -\int_\Omega u \operatorname{div}\omega\,dx \\
&= -\int_0^\infty dt \int_{u\geq t} \operatorname{div}\omega\,dx + \int_{-\infty}^0 dt \int_{u\leq t} \operatorname{div}\omega\,dx.
\end{aligned}$$

Since A_t is a smooth manifold, integrating by parts yields,

$$\int_{u\geq t} \operatorname{div}\omega \, dx = -\int_{u=t} \omega\nu \, d\mathcal{H}^{N-1} = -\int_{u=t} \omega\frac{\nabla u}{|\nabla u|} \, d\mathcal{H}^{N-1}$$

where ν is the normal to A_t directed into the set $\{x : \; u(x) \geq t\}$. Similarly,

$$\int_{u\leq t} \operatorname{div}\omega \, dx = \int_{u=t} \omega\nu \, d\mathcal{H}^{N-1} = \int_{u=t} \omega\frac{\nabla u}{|\nabla u|} \, d\sigma.$$

Making a change of variable in the last integral by letting $s := -t$, we obtain

$$\begin{aligned}\int_{\Omega} \omega\nabla u \, dx &= \int_0^{\infty} dt \int_{u=t} \omega\frac{\nabla u}{|\nabla u|} \, d\mathcal{H}^{N-1} + \int_0^{\infty} dt \int_{u=-t} \omega\frac{\nabla u}{|\nabla u|} \, d\mathcal{H}^{N-1} \\ &= \int_0^{\infty} dt \int_{A_t} \omega\frac{\nabla u}{|\nabla u|} \, d\mathcal{H}^{N-1}.\end{aligned}$$

1) Letting

$$\omega := \varphi\frac{\nabla u}{(|\nabla u|^2+\varepsilon)^{\frac{1}{2}}}$$

where $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$ and ε is a positive number, we obtain

$$\int_{\Omega} \varphi\frac{(\nabla u)^2}{((\nabla u)^2+\varepsilon)^{\frac{1}{2}}} \, dx = \int_0^{\infty} dt \int_{A_t} \varphi\frac{(\nabla u)^2}{|\nabla u|((\nabla u)^2+\varepsilon)^{\frac{1}{2}}} \, d\mathcal{H}^{N-1}.$$

Passing to the limit as $\varepsilon \downarrow 0$ and making use of Beppo Levi's Monotone Convergence Theorem we obtain (4.2) for all $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$.

2) Let $\varphi \in C(\Omega)$, $\varphi \geq 0$, $\operatorname{supp}[\varphi] \subset \Omega$ and let $\eta_\varepsilon\varphi$ be a **mollification** of φ (see [59, Section 2.7]) with radius ε. Since $\operatorname{supp}[\eta_\varepsilon\varphi] \subset \Omega$ for small values of ε, we have that

$$\int_{\Omega} (\eta_\varepsilon\varphi)\nabla u \, dx = \int_0^{\infty} dt \int_{A_t} \eta_\varepsilon\varphi \, d\mathcal{H}^{N-1}. \tag{4.3}$$

Let $\alpha \in \mathcal{D}(\Omega)$ satisfy

$$\alpha = 1 \text{ on } \bigcup_{\varepsilon} \operatorname{supp}[\eta_\varepsilon\varphi], \; \alpha \geq 0.$$

Obviously,

$$\int_{A_t} \eta_\varepsilon\varphi \, d\mathcal{H}^{N-1} \leq \|\eta_\varepsilon\varphi\|_\infty \int_{A_t} \alpha \, d\mathcal{H}^{N-1} \leq \|\varphi\|_\infty \int_{A_t} \alpha \, d\mathcal{H}^{N-1}. \tag{4.4}$$

By (4.2) applied to $\varphi = \alpha$, the integral in the right-hand side of (4.4) is a summable function on $(0,\infty)$. Since $\eta_\varepsilon\varphi \to \varphi$ uniformly and $\mathcal{H}^{N-1}(A_t \cap \operatorname{supp}[\alpha]) < \infty$ for almost all t, it follows that

$$\int_{A_t} \eta_\varepsilon\varphi \, d\mathcal{H}^{N-1} \to \int_{A_t} \varphi \, d\mathcal{H}^{N-1} \quad \text{as } \varepsilon \to 0$$

for almost all t. Now, Lebesgue's Dominated Convergence Theorem ensures the possibility of passing to the limit as $\varepsilon \to 0$ in (4.3) and we obtain

$$\int_\Omega \varphi \nabla u \, dx = \int_0^\infty dt \int_{A_t} \varphi \, d\mathcal{H}^{N-1}$$

for all $\varphi \in C(\Omega), \varphi \geq 0$ and $\text{supp}[\varphi] \subset \Omega$.

3) Further, we remove the restriction $\text{supp}[\varphi] \subset \Omega$. Let $\varphi \in C(\Omega), \varphi \geq 0$ and α_m be a sequence of nonnegative functions in $\mathcal{D}(\Omega)$ such that:

$$\bigcup_m \text{supp}[\alpha_m] = \Omega, \; 0 \leq \alpha_m \leq 1,$$
$$\alpha_m = 1 \;\text{ for }\; x \in \text{supp}[\alpha_{m-1}].$$

Then $\text{supp}[\alpha_m \varphi] \subset \Omega$ and

$$\int_\Omega \alpha_m \varphi |\nabla u| \, dx = \int_0^\infty dt \int_{A_t} \alpha_m \varphi \, d\mathcal{H}^{N-1}.$$

Since the sequence $(\alpha_m \varphi)$ does not decrease, by Beppo Levi's Theorem we may pass to the limit as $m \to \infty$ which completes the proof. □

Definition 4.1.5. *Let $\Omega \subset \mathbb{R}^N$ be an open set.*

a) A function $u \in L^1(\Omega)$ is of **bounded variation** *in Ω if*

$$\sup \left\{ \int_\Omega u \, \text{div} \, \varphi \, dx, \; \varphi \in C_c^1(\Omega, \mathbb{R}^N) : \; |\varphi| \leq 1 \right\} < \infty.$$

We write $BV(\Omega)$ to denote the space of functions of bounded variation.

b) An $\mathcal{L}^N$-measurable set $G \subset \mathbb{R}^N$ has a **finite perimeter** *in Ω if $\chi_G \in BV(\Omega)$.*

Next we give a version of the well-known Isoperimetric inequality. For the general case, we refer to [4, Theorem 3.46 p.149] or [73, Theorem 6.1.5 p.301].

Lemma 4.1.6 (Isoperimetric inequality). *Let G be a bounded measurable set of finite perimeter in $\mathbb{R}^N$. Then*

$$|G|^{\frac{N-1}{N}} \leq C(N) \mathcal{H}^{N-1}(\partial G). \tag{4.5}$$

The constant $C(N)$ is called the **isoperimetric constant** *and is given by*

$$C(N) := \frac{[\Gamma(\frac{N}{2}+1)]^{1/N}}{N\sqrt{\pi}}$$

where Γ denotes the usual gamma function.

Next for an open subset Ω of $\mathbb{R}^N$ and $1 \le p < \infty$ we denote by $W_p^1(\Omega)$ the space of all functions $u \in L^p(\Omega)$ such that $\nabla u \in L^p(\Omega)^N$ equipped with the norm

$$\|u\|_{W_p^1(\Omega)} := \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{1/p}.$$

Then $H^1(\Omega) = W_2^1(\Omega)$.

Now we are in position to give and to prove the Maz'ya inequality.

Theorem 4.1.7 (Maz'ya). *Let $\Omega \subset \mathbb{R}^N$ be an open set. For all $u \in W_1^1(\Omega) \cap C^\infty(\Omega) \cap C_c(\bar{\Omega})$, the inequality*

$$\|u\|_{\frac{N}{N-1}} \le C(N)(\|\nabla u\|_1 + \|u\|_{L^1(\partial\Omega,\sigma)}) \tag{4.6}$$

holds. Here $C(N)$ is the isoperimetric constant.

Proof. Let $u \in W_1^1(\Omega) \cap C^\infty(\Omega) \cap C_c(\bar{\Omega})$ and let

$$E_t := \{x \in \Omega : \ |u(x)| \ge t\} \quad \text{and} \quad A_t := \{x \in \Omega : \ |u(x)| = t\}.$$

Then E_t is a bounded set of finite perimeter in $\mathbb{R}^N$. By Lemma 4.1.6,

$$|E_t|^{\frac{N-1}{N}} \le C(N)\mathcal{H}^{N-1}(\partial E_t).$$

Here ∂E_t is the boundary of E_t in $\mathbb{R}^N$; i.e., $\partial E_t = A_t \cup (\bar{E}_t \cap \partial\Omega)$ where $\bar{E}_t$ is the closure of E_t in $\mathbb{R}^N$. We obtain that,

$$|E_t|^{\frac{N-1}{N}} \le C(N)\left(\mathcal{H}^{N-1}(A_t) + \mathcal{H}^{N-1}(\bar{E}_t \cap \partial\Omega)\right). \tag{4.7}$$

Thus

$$\begin{aligned}\int_0^\infty |E_t|^{\frac{N-1}{N}}\, dt &\le C(N)\left(\int_0^\infty \mathcal{H}^{N-1}(A_t)\, dt + \int_0^\infty \mathcal{H}^{N-1}(\bar{E}_t \cap \partial\Omega)\, dt\right)\\ &\le C(N)\left(\int_0^\infty dt \int_{A_t} d\mathcal{H}^{N-1} + \int_0^\infty dt \int_{\bar{E}_t\cap\partial\Omega} d\mathcal{H}^{N-1}\right).\end{aligned}$$

By Theorem 4.1.1, we have

$$\|u\|_{\frac{N}{N-1}} \le \int_0^\infty |E_t|^{\frac{N-1}{N}}\, dt. \tag{4.8}$$

Applying Theorem 4.1.4 with $\varphi = 1$ we obtain

$$\|\nabla u\|_1 = \int_0^\infty dt \int_{A_t} d\mathcal{H}^{N-1}. \tag{4.9}$$

Remark that $\bar{E}_t \cap \partial\Omega = \{x \in \partial\Omega : \ |u(x)| \ge t\}$. Let σ be the restriction to $\partial\Omega$ of the measure $\mathcal{H}^{N-1}$. Then applying Lemma 4.1.2 with the measure space $(\partial\Omega, \mathcal{B}(\partial\Omega), \sigma)$ we obtain that

$$\int_{\partial\Omega} |u|\, d\sigma = \int_0^\infty \sigma(\bar{E}_t \cap \partial\Omega)\, dt. \tag{4.10}$$

Now (4.8), (4.9) and (4.10) give (4.6). $\square$

Corollary 4.1.8. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then*

$$\|u\|_{\frac{2N}{N-1}} \leq C(N)^{1/2}\left(\|\nabla u\|_2^2 + \|u\|_2^2 + \|u\|^2_{L^2(\partial\Omega,\sigma)}\right)^{1/2} \tag{4.11}$$

for all $u \in H^1(\Omega) \cap C^\infty(\Omega) \cap C_c(\bar{\Omega})$.

Proof. Let $u \in H^1(\Omega) \cap C^\infty(\Omega) \cap C_c(\bar{\Omega})$. Then $u^2 \in W_1^1(\Omega) \cap C^\infty(\Omega) \cap C_c(\bar{\Omega})$. By Theorem 4.1.7, one has that

$$\begin{aligned}
\|u\|^2_{\frac{2N}{N-1}} = \|u^2\|_{\frac{N}{N-1}} &\leq C(N)\left(\|\nabla(u^2)\|_1 + \|u^2\|_{L^1(\partial\Omega,\sigma)}\right) \\
&\leq C(N)\left(2\|u\nabla u\|_1 + \|u\|^2_{L^2(\partial\Omega,\sigma)}\right) \\
&\leq C(N)\left(2\|\nabla u\|_2\|u\|_2 + \|u\|^2_{L^2(\partial\Omega,\sigma)}\right) \\
&\leq C(N)\left(\|\nabla u\|_2^2 + \|u\|_2^2 + \|u\|^2_{L^2(\partial\Omega,\sigma)}\right)
\end{aligned}$$

which is the inequality (4.11). □

We will call the inequality (4.11) **Maz'ya inequality for open sets with infinite measure**.

Corollary 4.1.9. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open set of finite measure. Then*

$$\|u\|_{\frac{2N}{N-1}} \leq C(N, |\Omega|)\left(\|\nabla u\|_2^2 + \|u\|^2_{L^2(\partial\Omega,\sigma)}\right)^{1/2} \tag{4.12}$$

for all $u \in H^1(\Omega) \cap C^\infty(\Omega) \cap C_c(\bar{\Omega})$.

Proof. Let $u \in H^1(\Omega) \cap C^\infty(\Omega) \cap C_c(\bar{\Omega})$. Then $u^2 \in W_1^1(\Omega) \cap C_c(\bar{\Omega}) \cap C^\infty(\Omega)$ and by Theorem 4.1.7, one obtains that

$$\begin{aligned}
\|u\|^2_{\frac{2N}{N-1}} = \|u^2\|_{\frac{N}{N-1}} &\leq C(N)\left(\|\nabla(u^2)\|_1 + \|u^2\|_{L^1(\partial\Omega,\sigma)}\right) \\
&\leq C(N)\left(2\|u\nabla u\|_1 + \|u\|^2_{L^2(\partial\Omega,\sigma)}\right).
\end{aligned} \tag{4.13}$$

By Hölder's inequality and the **Young inequality** $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2$ for all $a, b \geq 0$ and for every $\varepsilon > 0$, we obtain that

$$\begin{aligned}
2\|u\nabla u\|_1 \leq 2\|u\|_2\|\nabla u\|_2 &\leq 2|\Omega|^{\frac{1}{2N}}\|u\|_{\frac{2N}{N-1}}\|\nabla u\|_2 \\
&\leq \varepsilon\|u\|^2_{\frac{2N}{N-1}} + \frac{1}{\varepsilon}|\Omega|^{\frac{1}{N}}\|\nabla u\|_2^2
\end{aligned}$$

for every $\varepsilon > 0$. Choosing $\varepsilon := \frac{1}{2C(N)}$ and substituting the above inequality in (4.13) we obtain that

$$\|u\|^2_{\frac{2N}{N-1}} \leq 2C(N)\max\{1, 2|\Omega|^{\frac{1}{N}}C(N)\}\left(\|\nabla u\|_2^2 + \|u\|^2_{L^2(\partial\Omega,\sigma)}\right)$$

and the proof is complete. □

We will call the inequality (4.12) **Maz'ya inequality for open sets with finite measure**.

Definition 4.1.10. *Let $\Omega \subset \mathbb{R}^N$ be an open set. We denote by $W^1_{2,2}(\Omega, \partial\Omega)$ the completion of the set*

$$W_\sigma := \left\{ H^1(\Omega) \cap C^\infty(\Omega) \cap C_c(\bar{\Omega}) : \int_{\partial\Omega} |u|^2 \, d\sigma < \infty \right\}$$

with respect to the norm

$$|||u|||_1 := \|u\|_{W^1_{2,2}(\Omega,\partial\Omega)} := \left(\|\nabla u\|_2^2 + \|u\|_2^2 + \|u\|^2_{L^2(\partial\Omega,\sigma)} \right)^{1/2}. \tag{4.14}$$

If Ω has a finite measure then this norm is equivalent to the norm

$$|||u|||_2 := \left(\|\nabla u\|_2^2 + \|u\|^2_{L^2(\partial\Omega,\sigma)} \right)^{1/2}. \tag{4.15}$$

We call $W^1_{2,2}(\Omega, \partial\Omega)$ the **Maz'ya space**.

Remark that in both cases (Ω of finite measure or not), by Corollary 4.1.8, the space W_σ equipped with the norm defined by (4.14) satisfies

$$(W_\sigma, ||| \cdot |||_1) \hookrightarrow L^{\frac{2N}{N-1}}(\Omega) \tag{4.16}$$

where the embedding constant just depends on the dimension N.

If W_σ is equipped with the norm defined by (4.15), then by Corollary 4.1.9,

$$(W_\sigma, ||| \cdot |||_2) \hookrightarrow L^{\frac{2N}{N-1}}(\Omega) \tag{4.17}$$

for every open set with finite measure and in that case the embedding constant depends on N and an upper bound for $|\Omega|$. If Ω has an infinite measure, under some geometric conditions on Ω, the embedding (4.17) is sometimes true (see [73, Theorem 4.11.1.1]).

4.2 The Laplacian with Robin Boundary Conditions.

Throughout this section $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is an open set not necessarily of finite measure. Since

$$E := \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \int_{\partial\Omega} |u|^2 \, d\sigma < \infty\}$$

is a dense subspace of $W^1_{2,2}(\Omega, \partial\Omega)$ we can replace W_σ by E in the definition of $W^1_{2,2}(\Omega, \partial\Omega)$. Let us denote by j the natural embedding from E into $L^{\frac{2N}{N-1}}(\Omega)$. As a bounded linear operator, it has a unique extension to an operator

$$\tilde{\jmath} \in \mathcal{L}(W^1_{2,2}(\Omega, \partial\Omega), L^{\frac{2N}{N-1}}(\Omega)).$$

Robin boundary conditions have been studied by Daners [34]. In particular, he conjectured that the mapping $\tilde{\jmath}$ is always injective provided that $\sigma(\partial\Omega) < \infty$. We shall prove in this section that this conjecture is not true. Letting V denote the domain of the closure of the closable part of (a_σ, E), the results of Chapter 3 imply that $V = (\ker \tilde{\jmath})^\perp$. Therefore, the restriction of $\tilde{\jmath}$ to V is injective, and since V is a closed subspace of $W^1_{2,2}(\Omega, \partial\Omega)$, it follows that

$$V \hookrightarrow L^{\frac{2N}{N-1}}(\Omega). \tag{4.18}$$

More precisely, to see that (4.18) holds, let $\sigma = \sigma_r + \sigma_s$ be the decomposition of σ as in Lemma 3.3.5 or as in Remark 3.3.7. Note that E is dense in V. Recall that $\sigma_r = \chi_S \sigma$ and $\sigma_s = \chi_N \sigma$ for some relatively polar Borel set N and $S := \Gamma \setminus N$ where

$$\Gamma := \{z \in \partial\Omega : \ \exists\, r > 0 : \ \sigma(B(z, r) \cap \partial\Omega) < \infty\}.$$

Since σ_s is a regular Borel measure, there exists an increasing sequence of compact sets $K_n \subset N$ such that $\sigma_s(N \backslash \cup_n K_n) = 0$. Let $u \in E$. As in the proof of Proposition 3.3.8, we find a sequence $u_n \in E$ such that $u_n \to u$ in $\widetilde{H}^1(\Omega)$, $u_n|_{\partial\Omega} \to u|_{\partial\Omega}$ in $L^2(\partial\Omega, \sigma_r) = L^2(S, \sigma)$ and $u_n|_{\partial\Omega} \to 0$ in $L^2(N, \sigma)$.

If Ω has a finite measure, inserting the sequence u_n in the Maz'ya inequality (4.12) and taking the limit as $n \to \infty$, we obtain that

$$\|u\|_{\frac{2N}{N-1}} \le C(N, |\Omega|) \left(\|\nabla u\|_2^2 + \|u\|^2_{L^2(S,\sigma)} \right)^{1/2} = \|u\|_V. \tag{4.19}$$

If Ω has an infinite measure, inserting the sequence u_n in the Maz'ya inequality (4.11) and taking the limit as $n \to \infty$, we obtain that

$$\|u\|_{\frac{2N}{N-1}} \le C(N) \left(\|\nabla u\|_2^2 + \|u\|_2^2 + \|u\|^2_{L^2(S,\sigma)} \right)^{1/2} = \|u\|_V. \tag{4.20}$$

In both cases, we denote by Δ_R the selfadjoint operator on $L^2(\Omega)$ associated with the closure of the closable part of (a_σ, E). The operator Δ_R is **the Laplacian with Robin boundary conditions**. Using the inequalities (4.19) and (4.20) we will prove that the semigroup $(e^{t\Delta_R})_{t\ge0}$ on $L^2(\Omega)$ generated by Δ_R has a kernel which satisfies some Gaussian estimates with modified exponents, a result obtained by Daners [34] in the case where Ω is bounded.

A. The Embedding $W^1_{2,2}(\Omega, \partial\Omega)$ into $L^{\frac{2N}{N-1}}(\Omega)$ is not always Injective.

In this subsection, we prove that the form (a_σ, E) is not always closable.

Theorem 4.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain of finite measure. Let j be the natural embedding from E into $L^2(\Omega)$ and $\tilde{\jmath}$ its continuous extension from $W^1_{2,2}(\Omega, \partial\Omega)$ into $L^2(\Omega)$. Then $\tilde{\jmath}$ is not always injective.*

Proof. The results of Chapter 3 imply that the injectivity of $\tilde{\jmath}$ is equivalent to the closability of the form (a_σ, E) which is also equivalent to the fact that $\sigma \in \mathcal{M}_0$ in the case where $\sigma(\partial\Omega) < \infty$. Let $\Omega \subset \mathbb{R}^3$ be the domain constructed in Example 2.3.10. We have proved that for this domain, the 2-dimensional Hausdorff measure on $\partial\Omega$ charges relatively polar subsets of $\partial\Omega$; i.e., $\sigma \notin \mathcal{M}_0$. Since $\sigma(\partial\Omega) < \infty$, this implies that $\tilde{\jmath}$ is not injective. To illustrate this, replacing μ by σ in the proof of Theorem 3.3.1 ((i) $\Rightarrow$ (ii)), we have a sequence $w_k \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfying: w_k converges to zero in $H^1(\Omega)$, $w_k|_{\partial\Omega}$ is a Cauchy sequence in $L^2(\partial\Omega, \sigma)$ and $w_k|_{\partial\Omega}$ converges to χ_K pointwise where K is the subset of $\partial\Omega$ satisfying $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$ but $\sigma(K) > 0$. Since $w_k|_{\partial\Omega}$ is a Cauchy sequence in $L^2(\partial\Omega, \sigma)$ and converges to χ_K pointwise, the uniqueness of the limit implies that $w_k|_{\partial\Omega}$ converges to χ_K in $L^2(\partial\Omega, \sigma)$ but the function χ_K is not zero since $\sigma(K) > 0$. $\square$

Remark 4.2.2. *It follows from Proposition 2.3.4 and Theorem 2.1.5 that if $\widetilde{H}^1(\Omega)$ has the continuous extension property, then $\sigma \in \mathcal{M}_0$ and therefore (a_σ, E) is closable. In particular if Ω is a bounded Lipschitz domain then (a_σ, E) is closable and in that case, since each $u \in \widetilde{H}^1(\Omega)$ has a trace in $L^2(\partial\Omega, \sigma)$ and the trace application is continuous from $\widetilde{H}^1(\Omega)$ into $L^2(\partial\Omega, \sigma)$ (see below), it follows that $V = W^1_{2,2}(\Omega, \partial\Omega) = \widetilde{H}^1(\Omega) = H^1(\Omega)$.*

B. Characterization of the Domain of Δ_R.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Let

$$TH^1(\Omega) := \{f : \ f = u|_{\partial\Omega} : \ u \in H^1(\Omega)\}$$

be the **trace space** of $H^1(\Omega)$ equipped with the norm

$$\|f\|_{TH^1(\Omega)} := \inf\{\|u\|_{H^1(\Omega)} : \ u \in H^1(\Omega) : \ u|_{\partial\Omega} = f\}.$$

Let

$$H^{1/2}(\partial\Omega) := \{u \in L^2(\partial\Omega, \sigma) : \ \|u\|_{1/2,\partial\Omega} < \infty\},$$

where

$$\|u\|^2_{1/2,\partial\Omega} := \|u\|^2_{L^2(\partial\Omega,\sigma)} + \int_{\partial\Omega\times\partial\Omega\setminus d} \frac{|u(x) - u(y)|^2}{|x - y|^N} \, d(\sigma \otimes \sigma).$$

Then, it is clear that $H^{1/2}(\partial\Omega)$ is a Hilbert space. The following result due to Gagliardo is contained in [75, Theorem 4.1.1].

Theorem 4.2.3 (Gagliardo's Theorem). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Then the space $TH^1(\Omega)$ and $H^{1/2}(\partial\Omega)$ coincide with equivalent norms. Furthermore, there is a bounded linear extension operator*

$$\mathcal{E} : \ H^{1/2}(\partial\Omega) \to H^1(\mathbb{R}^N).$$

By Gagliardo's Theorem, the mapping

$$\gamma_0 : \ H^1(\Omega) \to H^{1/2}(\partial\Omega) : \ u \mapsto \gamma_0(u) := u|_{\partial\Omega}$$

is linear and for each $w \in H^{1/2}(\partial\Omega)$, there exists $v \in H^1(\Omega)$ such that $\gamma_0(v) = w$. Moreover there exists a constant $C > 0$ such that,

$$\|v\|_{H^1(\Omega)} \leq C\|w\|_{H^{1/2}(\partial\Omega)}. \tag{4.21}$$

Let

$$H(\Delta, \Omega) := \{u \in H^1(\Omega) : \ \Delta u \in L^2(\Omega)\}$$

be equipped with the norm

$$\|u\|^2_{H(\Delta,\Omega)} := \|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2_2.$$

Then $H(\Delta, \Omega)$ is a Hilbert space containing $C^\infty(\bar{\Omega})$ as a dense subspace (see [68, Theorem 8.2 p.39]). We will denote by $H^{-1/2}(\partial\Omega)$ the dual space of $H^{1/2}(\partial\Omega)$ where $\langle u, v\rangle = \int_{\partial\Omega} uv \, d\sigma$ if $u, v \in L^2(\partial\Omega, \sigma)$.

The proof of the following result given here is taken from [38, Chap. VII Lemma 2.2.1 and Corollary 2.2.1].

Lemma 4.2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Then the mapping $\gamma_1 : \ C^\infty(\bar{\Omega}) \to C(\partial\Omega)$ defined by $\gamma_1(u) := \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ (where ν denotes the exterior normal to $\partial\Omega$) has an extension, again denoted by γ_1, which is a continuous mapping of $H(\Delta, \Omega)$ into $H^{-1/2}(\partial\Omega)$. Moreover, for every $u \in H(\Delta, \Omega)$ and $v \in H^1(\Omega)$, we have the* **generalized Green formula**

$$\langle \gamma_1(u), v\rangle = \int_\Omega \Delta uv \, dx + \int_\Omega \nabla u \nabla v \, dx \tag{4.22}$$

where $\langle , \rangle$ denotes the duality between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

Proof. Let $u \in C^\infty(\bar{\Omega})$. For $w \in H^{1/2}(\partial\Omega)$, we let

$$l(w) := \int_{\partial\Omega} \frac{\partial u}{\partial \nu} w \, d\sigma.$$

Let $v \in H^1(\Omega)$ be such that $\gamma_0(v) = w$. Since Ω has a Lipschitz boundary, it follows that for $u \in C^\infty(\bar{\Omega})$ and $v \in H^1(\Omega)$, we have the Green formula

$$l(w) = \int_\Omega \Delta uv \, dx + \int_\Omega \nabla u \nabla v \, dx.$$

From Schwarz inequality and (4.21), we therefore have

$$l(w) \leq \|u\|_{H(\Delta,\Omega)}\|v\|_{H^1(\Omega)} \leq C\|u\|_{H(\Delta,\Omega)}\|w\|_{H^{1/2}(\partial\Omega)}$$

which proves that $l \in H^{-1/2}(\partial\Omega)$ and, in addition, that the mapping $\gamma_1 : u \in C^\infty(\bar{\Omega}) \to l \in H^{-1/2}(\partial\Omega)$ is bounded on $C^\infty(\bar{\Omega})$ equipped with the norm of $H(\Delta,\Omega)$. Since $C^\infty(\bar{\Omega})$ is dense in $H(\Delta,\Omega)$, we have that γ_1 can be extended by density to a continuous linear mapping from $H(\Delta,\Omega)$ into $H^{-1/2}(\partial\Omega)$.

Finally, since for each $u \in H(\Delta,\Omega)$, $\gamma_1(u)$ is a linear continuous functional on $H^{1/2}(\partial\Omega)$, we have the generalized Green formula (4.22). □

The following result characterizes the domain of the operator Δ_R.

Proposition 4.2.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Then the operator Δ_R is given by*

$$\begin{cases} D(\Delta_R) & = \{u \in H(\Delta,\Omega) : \left(\frac{\partial u}{\partial \nu} + u\right)|_{\partial\Omega} = 0\} \\ \Delta_R u & = \Delta u. \end{cases} \tag{4.23}$$

Proof. Let us note that $\left(\frac{\partial u}{\partial \nu} + u\right)|_{\partial\Omega} = 0$ means that $\left(\frac{\partial u}{\partial \nu} + u\right)|_{\partial\Omega} = 0$ in $H^{-1/2}(\partial\Omega)$; i.e., for all $\varphi \in H^{1/2}(\partial\Omega)$ we have $\langle\left(\frac{\partial u}{\partial \nu} + u\right)|_{\partial\Omega}, \varphi\rangle = 0$. Since Ω has a Lipschitz boundary, it follows that $\sigma \in \mathcal{M}_0$ (since $\widetilde{H}^1(\Omega) = H^1(\Omega)$ has the extension property) and $V = W^1_{2,2}(\Omega,\partial\Omega) = H^1(\Omega)$ with equivalent norm. By definition

$$\begin{cases} D(\Delta_R) & := \{u \in H^1(\Omega) : \exists\, v \in L^2(\Omega) : a_\sigma(u,\varphi) = (v,\varphi)\ \forall\, \varphi \in H^1(\Omega)\} \\ \Delta_R u & := -v. \end{cases}$$

Let

$$D := \{u \in H(\Delta,\Omega) : \left(\frac{\partial u}{\partial \nu} + u\right)|_{\partial\Omega} = 0\}.$$

We have to show that $D = D(\Delta_R)$. Let $u \in D$ and $v := -\Delta u$. Then $v \in L^2(\Omega)$. Using the generalized Green formula (4.22), we obtain that for all $\varphi \in H^1(\Omega)$,

$$(v,\varphi) := (-\Delta u,\varphi) = \int_\Omega \nabla u \nabla v\, dx - \langle\gamma_1(u),\varphi\rangle.$$

Since $u \in D$, it follows that

$$\langle\gamma_1(u) + \gamma_0(u),\varphi\rangle = 0.$$

This implies that

$$\langle\gamma_1(u),\varphi\rangle = -\langle\gamma_0(u),\varphi\rangle.$$

Therefore,

$$\begin{aligned}(v,\varphi)=\int_\Omega v\varphi\,dx := -\int_\Omega \Delta u\varphi\,dx &= \int_\Omega \nabla u\nabla\varphi\,dx+\langle\gamma_0(u),\varphi\rangle\\ &= \int_\Omega \nabla u\nabla\varphi\,dx+\int_{\partial\Omega} u\varphi\,d\sigma\\ &= a_\sigma(u,\varphi).\end{aligned}$$

Thus $u\in D(\Delta_R)$ and $\Delta_R u=-v=\Delta u$.

To prove the converse inclusion, let $u\in D(\Delta_R)$. Then, by definition, there exists $v\in L^2(\Omega)$ such that for all $\varphi\in H^1(\Omega)$ we have

$$a_\sigma(u,\varphi):=\int_\Omega \nabla u\nabla\varphi\,dx+\int_{\partial\Omega} u\varphi\,d\sigma=\int_\Omega v\varphi\,dx. \tag{4.24}$$

If we choose $\varphi\in\mathcal{D}(\Omega)$, (4.24) can be written

$$\langle -\Delta u,\varphi\rangle=\langle v,\varphi\rangle$$

where $\langle,\rangle$ denotes the duality between $\mathcal{D}(\Omega)'$ and $\mathcal{D}(\Omega)$. Since $\varphi\in\mathcal{D}(\Omega)$ is arbitrary, it follows that

$$-\Delta u=v \quad\text{in}\quad \mathcal{D}(\Omega)'.$$

As $v\in L^2(\Omega)$, this implies that $\Delta u\in L^2(\Omega)$ and then $u\in H(\Delta,\Omega)$. To finish, we have to prove that $\gamma_1(u)+\gamma_0(u)=0$ in $H^{-1/2}(\partial\Omega)$. Using the generalized Green formula, we obtain that for all $\varphi\in H^1(\Omega)$

$$\begin{aligned}\int_\Omega v\varphi\,dx &:= \int_\Omega(-\Delta u)\varphi\,dx=\int_\Omega \nabla u\nabla\varphi\,dx-\langle\gamma_1(u),\varphi\rangle\\ &= \int_\Omega \nabla u\nabla\varphi\,dx+\int_{\partial\Omega} u\varphi\,d\sigma-\int_{\partial\Omega} u\varphi\,d\sigma-\langle\gamma_1(u),\varphi\rangle\\ &= a_\sigma(u,\varphi)-\langle\gamma_1(u)+\gamma_0(u),\varphi\rangle.\end{aligned}$$

Since $u\in D(\Delta_R)$, it follows that $\int_\Omega v\varphi\,dx=a_\sigma(u,\varphi)$ and thus $\langle\gamma_1(u)+\gamma_0(u),\varphi\rangle=0$. Since $\varphi|_{\partial\Omega}\in H^{1/2}(\partial\Omega)$ is arbitrary, this implies that

$$\gamma_1(u)+\gamma_0(u)=0 \quad\text{in}\quad H^{-1/2}(\partial\Omega)$$

and thus $u\in D$ which completes the proof. □

Next we assume that $\Omega\subset\mathbb{R}^N$ is an arbitrary bounded domain. We simply denote by (a_σ,V) the closure of the closable part of (a_σ,E). Let S be the subset of $\partial\Omega$ obtained by decomposing σ as in Lemma 3.3.5 and let

$$V(\Delta,\Omega):=\{u\in V:\ \Delta u\in L^2(\Omega)\}$$

be equipped with the norm

$$\|u\|_{V(\Delta,\Omega)}^2 := \|u\|_V^2 + \|\Delta u\|_2^2.$$

Then $V(\Delta,\Omega)$ is a Hilbert space. We claim that for each $u \in V(\Delta,\Omega)$, we can define a "normal derivative" of u on S in the "generalized sense".

It is well-known (see Chapters 2 and 3) that $H_0^1(\Omega)$ is a closed subspace of V and by Theorem 2.4.1 it is given by

$$H_0^1(\Omega) = \{u \in V : \ \tilde{u} = 0 \ \text{ r.q.e. on } \ \partial\Omega\}.$$

Throughout the following, without restriction, we assume that each function $u \in \widetilde{H}^1(\Omega)$ is r.q.c.

Let $V(\partial\Omega) := V/H_0^1(\Omega)$ be equipped with the quotient norm

$$\|\dot{u}\|_{V(\partial\Omega)} := \inf_{v\in H_0^1(\Omega)} \|u-v\|_V := \inf_{v\in H_0^1(\Omega)} \Big(\|\nabla(u-v)\|_2^2 + \|u\|_{L^2(S,\sigma)}^2\Big)^{1/2}.$$

It is then clear that $V(\partial\Omega)$ is a Hilbert space and the mapping $\gamma_0 : \ V \to V(\partial\Omega) : u \mapsto \gamma_0(u) = u|_S$ is linear continuous. In particular, the mapping $\gamma_0 : \ V(\Delta,\Omega) \to V(\partial\Omega) : \ u \mapsto \gamma_0(u)$ is linear continuous. Let V' denotes the dual space of V. Since the mapping γ_0 defined above is linear continuous, it follows that the mapping L defined by: for $u \in V(\Delta,\Omega)$,

$$Lu : \ V \to \mathbb{R} : \ v \mapsto (Lu)v := \langle -\Delta u, v\rangle - a_\sigma(u,v),$$

where $\langle , \rangle$ denotes the duality between V' and V, defines a linear continuous functional on V. Moreover, since for each $v \in H_0^1(\Omega)$

$$\langle -\Delta u, v\rangle = \int_\Omega \nabla u \nabla v = a_\sigma(u,v),$$

it follows that the functional (Lu) is zero on $H_0^1(\Omega)$ and hence it defines a linear continuous functional on $V(\partial\Omega)$. By Riesz's Representation Theorem (see [24, Théorème IV.11]), there exists a unique element $G(u) \in V(\partial\Omega)'$ (the dual space of $V(\partial\Omega)$) such that for all $v \in V$,

$$\langle G(u), v\rangle_{V(\partial\Omega)'\times V(\partial\Omega)} = \langle -\Delta u, v\rangle_{V'\times V} - a_\sigma(u,v). \tag{4.25}$$

We define

$$\frac{\partial u}{\partial \nu} := -(G(u)+u) \ \text{ on } \ S \ \text{ in the "generalized sense"}$$

or

$$\frac{\partial u}{\partial \nu} := -(G(u)+u) \ \text{ in } \ V(\partial\Omega)'$$

and we call $\frac{\partial u}{\partial \nu}$ the **normal derivative** of $u \in V(\Delta,\Omega)$ on S in the "generalized sense".

Remark 4.2.6. *The notion of the "normal derivative" of a function u in $V(\Delta, \Omega)$ defined above is only formal since for an arbitrary domain the normal does not always exists σ a.e.*

Next, since the embedding from $V(\partial\Omega)$ into $L^2(S,\sigma)$ is continuous, it follows that every $g \in L^2(S,\sigma)$ defines a linear continuous functional on $V(\partial\Omega)$. Therefore, we can write

$$\langle G(u), v\rangle_{V(\partial\Omega)'\times V(\partial\Omega)} = \int_S gv\, d\sigma \;\; \forall\, v \in V;$$

that is $G(u) = g$ on S in the "generalized sense" or $G(u) = g$ in $V(\partial\Omega)'$.

The following result characterizes the domain of the operator Δ_R for an arbitrary bounded domain as in the case where Ω has a Lipschitz boundary.

Proposition 4.2.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then the operator Δ_R is given by*

$$\begin{cases} D(\Delta_R) & = \{u \in V(\Delta, R) : \; G(u) = 0 \;\; on \;\; S\} \\ \Delta_R u & = \Delta u. \end{cases}$$

Proof. By definition, the operator Δ_R is given by

$$\begin{cases} D(\Delta_R) & := \{u \in V : \; \exists\, v \in L^2(\Omega) : \; a_\sigma(u,\varphi) = (v,\varphi) \;\forall\, \varphi \in V\} \\ \Delta_R u & := -v. \end{cases}$$

Let

$$D := \{u \in V(\Delta,\Omega) : \; G(u) = 0 \;\; \text{on} \;\; S\}.$$

Let $u \in D$ and $v := -\Delta u$. Then $v \in L^2(\Omega)$. Using (4.25), we have

$$(v,\varphi) := (-\Delta u, \varphi) = \langle -\Delta u, \varphi\rangle = a_\sigma(u,\varphi) \;\; \forall\, \varphi \in V$$

and thus $u \in D(\Delta_R)$ and $\Delta_R u = -v = \Delta u$.

To prove the converse inclusion, let $u \in D(\Delta_R)$. By definition, there exists $v \in L^2(\Omega)$ such that for all $\varphi \in V$ we have

$$\int_\Omega \nabla u \nabla\varphi\, dx + \int_S u\varphi\, d\sigma = \int_\Omega v\varphi. \tag{4.26}$$

As in the proof of Proposition 4.2.5, the equality (4.26) implies that $\Delta u \in L^2(\Omega)$ and thus $u \in V(\Delta,\Omega)$. Finally, by (4.25), we obtain that for all $\varphi \in V$,

$$\int_\Omega v\varphi\, dx := \int_\Omega (-\Delta u)\varphi\, dx = \langle -\Delta u, \varphi\rangle_{V'\times V} = \langle G(u), \varphi\rangle_{V(\partial\Omega)'\times V(\partial\Omega)} + a_\sigma(u,\varphi).$$

Since $u \in D(\Delta_R)$, we have that

$$a_\sigma(u,\varphi) = \int_\Omega v\varphi\, dx$$

and thus

$$\langle G(u), \varphi \rangle_{V(\partial\Omega)' \times V(\partial\Omega)} = 0.$$

Since $\varphi|_S \in V(\partial\Omega)$ is arbitrary, this implies that $G(u) = 0$ in $V(\partial\Omega)'$. Thus $u \in D$ and the proof is complete. $\square$

Next we give a very important remark.

Remark 4.2.8. *Recall that the Laplacian with Robin boundary conditions is the closed operator Δ_R associated with the closed form (a_σ, V) where σ is the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure. We show that this classical definition has some disadvantages and thus seems not to be the correct definition.*

Let $\Omega \subset \mathbb{R}^2$ be the domain bounded by the von Koch curve as in Figure 4.1 It is well-known that σ is locally infinite on $\partial\Omega$ and thus the results of Chapter 3 imply that the closed form (a_σ, V) is given by $V = H_0^1(\Omega)$ and

$$a_\sigma(u,v) = \int_\Omega \nabla u \nabla v \, dx.$$

Thus the operator Δ_R is the Laplacian with Dirichlet boundary conditions.

By [73, Section 1.5.1 Example 1], the boundary $\partial\Omega$ of Ω is a quasicircle and then by [63, Theorem B and Theorem 4], $H^1(\Omega)$ has the extension property. Since Ω is a domain in $\mathbb{R}^2$, by Remark 2.3.6 $H^1(\Omega)$ has the continuous extension property. Let s be the Hausdorff dimension of $\partial\Omega$. By [46, Example 2.7 p.31] and [46, Section 9.2 p.117],

$$1 < s = \frac{\log 4}{\log 3} < 2 \quad \textit{and} \quad 0 < \mathcal{H}^s(\partial\Omega) < \infty.$$

Since $\widetilde{H}^1(\Omega) = H^1(\Omega)$ has the continuous extension property, by Proposition 2.3.4 and Theorem 2.1.5, $\mathcal{H}^s \in \mathcal{M}_0$. Let us still denote by $\mathcal{H}^s$ the restriction to $\partial\Omega$ of $\mathcal{H}^s$. Then the form $(a_{\mathcal{H}^s}, E)$ is closable on $L^2(\Omega)$ and its closure is given by:

$$a_{\mathcal{H}^s}(u,v) = \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \tilde{u}\tilde{v} \, d\mathcal{H}^s$$

with domain

$$V = \{u \in \widetilde{H}^1(\Omega) = H^1(\Omega) : \ \tilde{u} \in L^2(\partial\Omega, \mathcal{H}^s)\}.$$

Thus, we see that in order to define the Robin boundary conditions, it is more natural to take as measure the restriction of the s-dimensional Hausdorff measure to $\partial\Omega$ where s is the Hausdorff dimension of the boundary, than taking the measure σ. After this investigation, we can ask the following question.

Is the Maz'ya inequality also true if we consider the measure $\mathcal{H}^s$ in place of σ?

More precisely, let $\Omega \subset \mathbb{R}^N$ *be an open set with finite Lebesgue measure and* s *be the Hausdorff dimension of its boundary. Let* $\widetilde{W}$ *be the completion of the space*

$$E_{\mathcal{H}^s} := \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \int_{\partial\Omega} |u|^2 \, d\mathcal{H}^s < \infty\}$$

with respect to the norm

$$\|u\|^2_{\widetilde{W}} := \|\nabla u\|_2^2 + \|u\|^2_{L^2(\partial\Omega, \mathcal{H}^s)}.$$

Is it always possible to find $q > 2$ *such that the space* $\widetilde{W}$ *is continuously embedded into* $L^q(\Omega)$*?*

Figure 4.1: von Koch curve

C. Kernel Estimates and p-Independence of the Spectrum.

The following inequality called Nash inequality is only a consequence of the two Maz'ya inequalities. The proof given here is very easy and works also for an arbitrary Dirichlet space $D(a)$ which embeds into $L^p(X, m)$ for some $p > 2$ where X and m satisfy the condition (1.3).

Proposition 4.2.9 (Nash inequality). *There exists a constant* $c > 0$ *such that*

$$\|u\|_2^{2+\frac{2}{N}} \leq c\|u\|_V^2 \|u\|_1^{\frac{2}{N}} \tag{4.27}$$

for all $u \in V \cap L^1(\Omega)$. The constant c depends only on N if $|\Omega| = \infty$ and it depends on N and $|\Omega|$ if $|\Omega| < \infty$.

Proof. Let $u \in V \cap L^1(\Omega)$. Using the Hölder inequality, one obtains the following interpolation inequality:

$$\|u\|_2 \le \|u\|_1^{\frac{1}{N+1}} \|u\|_{\frac{2N}{N-1}}^{\frac{N}{N+1}}. \tag{4.28}$$

Using the Maz'ya inequality (4.11) if $|\Omega| = \infty$ and (4.12) if $|\Omega| < \infty$, one has that

$$\|u\|_{\frac{2N}{N-1}}^{\frac{N}{N+1}} \le c_1 \|u\|_V^{\frac{N}{2(N+1)}}. \tag{4.29}$$

Finally, replacing (4.29) in (4.28) we obtain that

$$\|u\|_2 \le c_1 \|u\|_1^{\frac{1}{N+1}} \|u\|_V^{\frac{N}{2(N+1)}}.$$

Thus

$$\|u\|_2^{2+\frac{2}{N}} \quad \le \quad c\|u\|_V^2 \|u\|_1^{\frac{2}{N}}.$$

□

The Nash inequality implies that the operator $e^{t\Delta_R}$ has a kernel. To obtain an estimate of this kernel, we shall use a technique of Arendt and Ter Elst. They have proved in [16] **Gaussian estimates** of semigroups generated by elliptic differential operators with general boundary conditions. Then the proof given here is only an adaptation of the proof contained in [16, Theorem 4.4]. Finally, we note that this technique uses Davies' perturbation method which is contained in [39].

Next, let

$$W := \{\psi \in C_b^\infty(\mathbb{R}^N) : \ \|\nabla\psi\|_\infty \le 1, \ \|D_i D_j \psi\|_\infty \le 1 \ \forall\, 1 \le i, j \le N\}.$$

For a semigroup T on $L^2(\Omega)$, $\rho \in \mathbb{R}$ and $\psi \in W$ we define the perturbed semigroup T_ρ on $L^2(\Omega)$ by $T_\rho(t) = U_\rho T(t) U_\rho^{-1}$ where $(U_\rho\varphi)(x) = e^{-\rho\psi(x)}\varphi(x)$.

To obtain the Gaussian estimates, we need the following result which is contained in [16, Proposition 3.3].

Proposition 4.2.10. *Let T be a semigroup on $L^2(\Omega)$ and $c, \omega_1 \in \mathbb{R}$. Then the following assertions are equivalent.*

(i) There exists a constant $\omega_2 > 0$ such that

$$\|T_\rho(t)\|_{1\to\infty} \le ct^{-N/2} e^{\omega_1 t + \omega_2 \rho^2 t}$$

uniformly for all $\rho \in \mathbb{R}, t > 0$ and $\psi \in W$.

(ii) *There exists a constant $b > 0$ such that the operators $T(t)$ have a kernel $K_t \in L^\infty(\Omega \times \Omega)$ which verifies*

$$|K_t(x,y)| \leq ct^{-N/2} e^{-b\frac{|x-y|^2}{t}} e^{\omega_2 t}$$

for (x,y) a.e. and for all $t > 0$.

The following is the main result of this subsection.

Theorem 4.2.11. *There exists a constant $C > 0$ depending on the constant of the inequality (4.27) such that*

$$\|e^{t\Delta_R}\|_{1\to\infty} \leq Ct^{-N} \tag{4.30}$$

uniformly for all $t > 0$ if $|\Omega| < \infty$.

There exists a constant $C > 0$ depending only on N such that

$$\|e^{t\Delta_R}\|_{1\to\infty} \leq Ce^t t^{-N} \tag{4.31}$$

uniformly for all $t > 0$ if $|\Omega| = \infty$. Moreover, the operators $e^{t\Delta_R}$ have a kernel $K_t \in L^\infty(\Omega \times \Omega)$ satisfying

$$0 \leq K_t(x,y) \leq Ct^{-N} e^{-b\frac{|x-y|^2}{t}} \tag{4.32}$$

for some constant $b > 0$, and for (x,y) a.e. and uniformly for all $t > 0$ if $|\Omega| < \infty$ and

$$0 \leq K_t(x,y) \leq Ct^{-N} e^{\omega t} e^{-b\frac{|x-y|^2}{t}} \tag{4.33}$$

for some constants $b, \omega > 0$, and for (x,y) a.e. and uniformly for all $t > 0$ if $|\Omega| = \infty$.

Proof. 1) First, we prove the inequality (4.30). Let $0 \leq f \in L^1(\Omega) \cap L^2(\Omega) = L^2(\Omega)$ and let

$$f_t := e^{t\Delta_R} f \quad \text{and} \quad u(t) := \|f_t\|_2^2.$$

Then $f_t \in V \cap L^1(\Omega) = V$. As $(e^{t\Delta_R})_{t\geq 0}$ is a holomorphic semigroup on $L^2(\Omega)$ then

$$-\frac{du}{dt} = -(\Delta_R f_t, f_t) = a_\sigma(f_t, f_t).$$

Using the inequality (4.27), one has

$$-\frac{du}{dt} = a_\sigma(f_t, f_t) \geq \frac{1}{c}\|f_t\|_2^{2+\frac{2}{N}} \|f_t\|_1^{\frac{-2}{N}} = \frac{1}{c} u(t)^{1+\frac{1}{N}} \|f_t\|_1^{\frac{-2}{N}}.$$

Therefore

$$\begin{aligned} \frac{d}{dt}\left(u(t)^{\frac{-1}{N}}\right) &= \frac{-1}{N} u(t)^{-\frac{1}{N}-1} \frac{d}{dt} u(t) \\ &\geq \frac{1}{N} u(t)^{-\frac{1}{N}-1} \frac{1}{c} u(t)^{\frac{1}{N}+1} \|f_t\|_1^{\frac{-2}{N}} \\ &\geq \frac{1}{cN} \|f_t\|_1^{\frac{-2}{N}}. \end{aligned} \tag{4.34}$$

Since $(e^{t\Delta_R})_{t\geq 0}$ is a submarkovian semigroup, one has the bound $\|f_t\|_1 \leq \|f\|_1$ and therefore

$$\|f_t\|_1^{-\frac{2}{N}} \geq \|f\|_1^{-\frac{2}{N}}. \tag{4.35}$$

Integrating (4.34) and using (4.35), we obtain

$$\int_0^t \frac{d}{ds}\left(u(s)^{\frac{-1}{N}}\right) ds = u(t)^{-\frac{1}{N}} - u(0)^{-\frac{1}{N}} \geq \frac{t}{cN}\|f\|_1^{-\frac{2}{N}}. \tag{4.36}$$

Since $u(0) := \|f\|_2^2 \geq 0$, the inequality (4.36) implies that

$$u(t)^{\frac{1}{N}} \leq \frac{cN}{t}\|f\|_1^{\frac{2}{N}}.$$

Finally,

$$\|e^{t\Delta_R} f\|_2 \leq \left(\frac{cN}{t}\right)^{\frac{N}{2}} \|f\|_1 = c_1 t^{\frac{-N}{2}} \|f\|_1.$$

This bound extends to all $0 \leq f \in L^1(\Omega)$ by an approximation argument and then to all $f \in L^1(\Omega)$ by the positivity of $e^{t\Delta_R}$. By duality, we have

$$\|e^{t\Delta_R} f\|_\infty \leq c_2 t^{-\frac{N}{2}} \|f\|_2.$$

Since $e^{t\Delta_R} = e^{\frac{t}{2}\Delta_R} e^{\frac{t}{2}\Delta_R}$, we have

$$\|e^{t\Delta_R}\|_{1\to\infty} \leq \|e^{t/2\Delta_R}\|_{1\to 2}\|e^{t/2\Delta_R}\|_{2\to\infty} \leq Ct^{-N}$$

uniformly for all $t > 0$.

2) Next we prove the inequality (4.31). This is the case where $|\Omega| = \infty$. For $f \in L^1(\Omega) \cap L^2(\Omega)$, we put

$$f_t := e^{t(\Delta_R - I)} f \quad\text{and}\quad u(t) := \|f_t\|_2^2.$$

Then $f_t \in V \cap L^1(\Omega)$. Proceeding exactely as in 1), we obtain that

$$\|e^{t(\Delta_R - I)}\|_{1\to\infty} \leq Ct^{-N}$$

uniformly for all $t > 0$. Thus

$$\|e^{t\Delta_R}\|_{1\to\infty} \leq Ct^{-N} e^t$$

uniformly for all $t > 0$.

3) Next we prove the bounds (4.32) and (4.33). By (4.30), Dunford-Pettis' Theorem (see [11, Theorem 1.2.6]) ensures that $e^{t\Delta_R}$ is an integral operator (see Definition 4.3.2 below) with kernel $K_t \in L^\infty(\Omega \times \Omega)$ for all $t > 0$. Recall that we assume that each function $u \in V$ is r.q.c., and

$$\|u\|_V^2 = \|\nabla u\|_2^2 + \|u\|_{L^2(S,\sigma)}^2$$

if $|\Omega| < \infty$ and

$$\|u\|_V^2 = \|\nabla u\|_2^2 + \|u\|_2^2 + \|u\|_{L^2(S,\sigma)}^2$$

if $|\Omega| = \infty$. For each $\rho \in \mathbb{R}$ and $\psi \in W$ we define the perturbed semigroup $T_\rho(t)$ on $L^2(\Omega)$ by $T_\rho(t) := U_\rho e^{t\Delta_R} U_\rho^{-1}$. This definition has a sense since $e^{-\rho\psi}\varphi \in L^p(\Omega)$ if $\varphi \in L^p(\Omega)$ for all $1 \le p \le \infty$.

a) Let $\varphi \in V$, then $U_\rho \varphi \in V$. Indeed, there exists a sequence $\varphi_n \in E$ such that $\varphi_n \to \varphi$ in V. It is clear that $U_\rho \varphi_n \in E$ and,

$$\begin{aligned}
\|\nabla(U_\rho\varphi_n - U_\rho\varphi)\|_2 &= \|\nabla(e^{-\rho\psi}(\varphi_n - \varphi))\|_2 \\
&\le \|e^{-\rho\psi}\nabla(\varphi_n - \varphi)\|_2 + \|\rho(\varphi_n - \varphi)e^{-\rho\psi}\nabla\psi\|_2 \\
&\le \|e^{-\rho\psi}\|_\infty \|\nabla(\varphi_n - \varphi)\|_2 + \|\rho e^{-\rho\psi}\|_\infty \|\varphi_n - \varphi\|_2 \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|U_\rho\varphi_n - U_\rho\varphi\|_2 &= \|e^{-\rho\psi}(\varphi_n - \varphi)\|_2 \\
&\le \|e^{-\rho\psi}\|_\infty \|\varphi_n - \varphi\|_2 \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|U_\rho\varphi_n - U_\rho\varphi\|_{L^2(S,\sigma)} &\le \|e^{-\rho\psi}\|_\infty \|\varphi_n - \varphi\|_{L^2(S,\sigma)} \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus $U_\rho \varphi \in V$.

b) Let A_ρ be the generator of the semigroup T_ρ. The form a_ρ associated with the operator A_ρ is given by

$$a_\rho : V \times V \to \mathbb{R}; \qquad a_\rho(u,v) = a_\sigma(U_\rho^{-1}u, U_\rho v).$$

Letting $\psi_i := D_i\psi$, a simple calculation gives

$$a_\rho(u,v) = \sum_{i=1}^N \int_\Omega (D_i + \rho\psi_i)u(D_i - \rho\psi_i)v\, dx + \int_S uv\, d\sigma.$$

Let $u \in V$, then

$$\begin{aligned}
a_\rho(u,u) &= \sum_{i=1}^N \int_\Omega (D_i + \rho\psi_i)u(D_i - \rho\psi_i)u\, dx + \int_S |u|^2\, d\sigma \\
&= \int_\Omega |\nabla u|^2\, dx - \rho^2 \int_\Omega |\nabla\psi|^2 |u|^2\, dx + \int_S |u|^2\, d\sigma.
\end{aligned}$$

Since

$$\rho^2 \int_\Omega |\nabla\psi|^2 |u|^2\, dx \le \rho^2 \int_\Omega |u|^2\, dx,$$

it follows that

$$a_\rho(u,u) + \rho^2\|u\|_2^2 \geq 0$$

which implies that

$$\|T_\rho(t)\|_{2\to 2} \leq e^{\rho^2 t}$$

for all $t > 0$.

c) Let $p \in 2\mathbb{N}$ and $\varphi \in L^2(\Omega) \cap L^\infty(\Omega)$. Then $(T_\rho(t)\varphi)^p \in V$ whenever $t > 0$. In fact, let $f = T_\rho(t)\varphi$. Then $f \in V$ and therefore $f \in V \cap L^\infty(\Omega)$. We show that $f^p \in V$. For $p = 2$, since $f^2 = f \cdot f$ and $f \in V \cap L^\infty(\Omega)$, by [55, Theorem 1.4.2 (ii)], $f^2 \in V$. Now, by induction on p, one obtains easily that $f^p \in V$.

d) Let $\varphi \in L^2(\Omega) \cap L^\infty(\Omega)$, $p \in 2\mathbb{N}$ and $\varphi_t := T_\rho(t)\varphi$. We claim that the mapping $t \mapsto \|T_\rho(t)\varphi\|_{2p}^{2p}$ is differentiable on $(0,\infty)$ and

$$\frac{d}{dt}\|T_\rho(t)\varphi\|_{2p}^{2p} = -2p(A_\rho\varphi_t, \varphi_t^{2p-1}) = -2pa_\rho(\varphi_t, \varphi_t^{2p-1}). \tag{4.37}$$

Indeed, consider the following mappings:

$$\begin{array}{rcl} G & : & (0,\infty) \to L^2(\Omega) \setminus \{0\} : \quad t \mapsto \varphi_t^p \\ F & : & L^2(\Omega) \setminus \{0\} \to \mathbb{R}_+ : \quad u \mapsto \|u\|_2^2. \end{array}$$

Clearly, $\|T_\rho(t)\varphi\|_{2p}^{2p} = F \circ G(t)$. Since T_ρ is a holomorphic semigroup on $L^2(\Omega)$, it follows that G is differentiable and

$$G'(t) = -p\varphi_t^{p-1}A_\rho\varphi_t.$$

It is also easy to verify that F is differentiable and for every $u \in L^2(\Omega) \setminus \{0\}$

$$F'(u) \cdot h = 2(u,h).$$

Thus the mapping $t \mapsto \|T_\rho(t)\varphi\|_{2p}^{2p}$ is differentiable on $(0,\infty)$ and

$$\begin{array}{rcl} \dfrac{d}{dt}\|T_\rho(t)\varphi\|_{2p}^{2p} & = & (F \circ G)'(t) = F'(G(t)) \cdot G'(t) \\ & = & -2p(A_\rho\varphi_t, \varphi_t^{2p-1}) = -2pa_\rho(\varphi_t, \varphi_t^{2p-1}) \end{array}$$

which proves the claim.

e) Next we show that there exists a constant $c > 0$ such that

$$\frac{d}{dt}\|T_\rho(t)\varphi\|_{2p}^{2p} \leq -\|\varphi_t^p\|_V + cp^2\rho^2\|\varphi_t^p\|_2^2$$

uniformly for all $t > 0, \rho \in \mathbb{R}, \psi \in W, \varphi \in V \cap L^\infty(\Omega)$ and $p \in 2\mathbb{N}$. By (4.37), a simple calculation gives

$$\begin{aligned}
\frac{d}{dt}\|T_\rho(t)\varphi\|_{2p}^{2p} &= -2p\left(\sum_{i=1}^N \int_\Omega (D_i+\rho\psi_i)\varphi_t(D_i-\rho\psi_i)\varphi_t^{2p-1}\,dx + \int_S |\varphi_t^p|^2\,d\sigma\right)\\
&= -2p\sum_{i=1}^N \int_\Omega D_i\varphi_t D_i\varphi_t^{2p-1}\,dx - 2p\sum_{i=1}^N\int_\Omega \rho\psi_i\varphi_t D_i\varphi_t^{2p-1}\,dx\\
&+ 2p\sum_{i=1}^N\int_\Omega \rho\psi_i\varphi_t^{2p-1}D_i\varphi_t\,dx + 2p\sum_{i=1}^N\int_\Omega \rho^2\psi_i^2\varphi_t^{2p}\,dx\\
&- 2p\|\varphi_t^p\|_{L^2(S,\sigma)}.
\end{aligned}$$

The first term can be estimated by

$$-2p\sum_{i=1}^N\int_\Omega D_i\varphi_t D_i\varphi_t^{2p-1}\,dx = \frac{-2p(2p-1)}{p^2}\|\nabla\varphi_t^p\|_2^2 \le -2\|\nabla\varphi_t^p\|_2^2.$$

In the following estimates, we shall use frequently the Hölder inequality and the Young inequality. The second term can be estimated by

$$\begin{aligned}
\left|-2p\sum_{i=1}^N\int_\Omega \rho\psi_i\varphi_t D_i\varphi_t^{2p-1}\,dx\right| &= \left|-2(2p-1)\rho\sum_{i=1}^N\int_\Omega \psi_i\varphi_t^p D_i\varphi_t^p\,dx\right|\\
&\le 4Np|\rho|\|\nabla\varphi_t^p\|_2\|\varphi_t^p\|_2\\
&\le \varepsilon\|\nabla\varphi_t^p\|_2^2 + \varepsilon^{-1}N^2\rho^2p^2\|\varphi_t^p\|_2^2
\end{aligned}$$

for every $\varepsilon>0$. The third term can be estimated by

$$\begin{aligned}
\left|2p\sum_{i=1}^N\int_\Omega \rho\psi_i\varphi_t^{2p-1}D_i\varphi_t\,dx\right| &\le 2|\rho|\left|\sum_{i=1}^N\int_\Omega \varphi_t^p D_i\varphi_t^p\,dx\right|\\
&\le 4Np|\rho|\|\nabla\varphi_t^p\|_2\|\varphi_t^p\|_2\\
&\le \varepsilon\|\nabla\varphi_t^p\|_2^2 + \varepsilon^{-1}N^2\rho^2p^2\|\varphi_t^p\|_2^2
\end{aligned}$$

for every $\varepsilon>0$. The fourth term can be estimated by

$$2p\rho^2\sum_{i=1}^N\int_\Omega \psi_i^2\varphi_t^{2p}\,dx \le 2N^2p^2\rho^2\|\varphi_t^p\|_2^2.$$

Finally, one has

$$\frac{d}{dt}\|T_\rho(t)\varphi\|_{2p}^{2p} \le -(2-2\varepsilon)\|\nabla\varphi_t^p\|_2^2 - \|\varphi_t^p\|_{L^2(S,\sigma)} + p^2\rho^2N^2(2+2\varepsilon^{-1})\|\varphi_t^p\|_2^2.$$

Choosing $\varepsilon = 1/2$, one obtains that

$$\frac{d}{dt}\|T_\rho(t)\varphi\|_{2p}^{2p} \le -\|\varphi_t^p\|_V + 6N^2p^2\rho^2\|\varphi_t^p\|_2^2$$

if $|\Omega| < \infty$ and

$$\frac{d}{dt}\|T_\rho(t)\varphi\|_{2p}^{2p} \leq -\|\varphi_t^p\|_V + 6N^2p^2(\rho^2+1)\|\varphi_t^p\|_2^2$$

if $|\Omega| = \infty$. Since $\|T_\rho(t)\|_{2\to 2} \leq e^{\rho^2 t}$ uniformly for all $t > 0, \rho \in \mathbb{R}$ and $\psi \in W$, applying [16, Proposition 4.6], we obtain that

$$\|T_\rho(t)\|_{2\to\infty} \leq ct^{-\frac{N}{2}}e^{\rho^2 t}e^{3N^2\rho^2 t} = ct^{-\frac{N}{2}}e^{(1+3N^2)\rho^2 t}$$

if $|\Omega| < \infty$ and

$$\|T_\rho(t)\|_{2\to\infty} \leq ct^{-\frac{N}{2}}e^{\rho^2 t}e^{3N^2(1+\rho^2)t} = ct^{-\frac{N}{2}}e^{3N^2 t}e^{(1+3N^2)\rho^2 t}$$

if $|\Omega| = \infty$. By duality we obtain

$$\|T_\rho(t)\|_{1\to 2} \leq ct^{-\frac{N}{2}}e^{(1+3N^2)\rho^2 t}$$

if $|\Omega| < \infty$ and

$$\|T_\rho(t)\|_{1\to 2} \leq ct^{-\frac{N}{2}}e^{3N^2 t}e^{(1+3N^2)\rho^2 t}$$

if $|\Omega| = \infty$. Hence

$$\|T_\rho(t)\|_{1\to\infty} \leq ct^{-N}e^{(1+3N^2)\rho^2 t}$$

for all $t > 0$ and $\rho \in \mathbb{R}$ if $|\Omega| < \infty$, and

$$\|T_\rho(t)\|_{1\to\infty} \leq ct^{-N}e^{3N^2 t}e^{(1+3N^2)\rho^2 t}$$

for all $t > 0$ and $\rho \in \mathbb{R}$ if $|\Omega| = \infty$. Now the claim follows by applying Proposition 4.2.10. $\square$

Now, since $(e^{t\Delta_R})$ is a submarkovian semigroup on $L^2(\Omega)$, by Theorem 1.3.17, there exist consistent semigroups on all $L^p(\Omega)$, $1 \leq p < \infty$. We denote by Δ_R^p the generator of the semigroup on $L^p(\Omega)$ for $1 \leq p < \infty$. Notice that $\Delta_R^2 = \Delta_R$

Proposition 4.2.12. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then the spectrum of Δ_R^p is independent of p; i.e., $\sigma(\Delta_R^p) = \sigma(\Delta_R)$ for all $1 \leq p < \infty$.*

The proof uses the following result due to Kunstmann and Vogt and contained in [67, Proposition 5] which can be reformulated as follows.

Theorem 4.2.13 (Kunstmann-Vogt). *Let $\Omega \subset \mathbb{R}^N$ be an open set and $(e^{tA})_{t\geq 0}$ be a submarkovain C_0-semigroup on $L^2(\Omega)$ such that e^{tA} has a kernel K_t satisfying the upper bound*

$$|K_t(x,y)| \leq c_1 t^{-M/2}e^{-c_2\frac{|x-y|^2}{t}} \qquad (t > 0,\ x,y \in \Omega)$$

for some constants $c_1, c_2 > 0$ and $M > N$. Let A_p be the generator of the semigroups on $L^p(\Omega)$ for $1 \leq p < \infty$. Then $\sigma(A_p) = \sigma(A_2) := \sigma(A)$.

In fact, by [67, Proposition 5], $R(\lambda, A_p)^n$ is independent of p for all large λ and for all $n \geq 1 + \frac{M-N}{2}$ and by [64, Lemma 6.3] we obtain that $\sigma(A_p) = \sigma(A_2)$.

Proof of Proposition 4.2.12. By Theorem 4.2.11, the semigroup $(e^{t\Delta_R})_{t\geq 0}$ is **ultracontractive**; i.e. $e^{t\Delta_R}$ maps $L^2(\Omega)$ into $L^\infty(\Omega)$ (see [39, Section 2.1]).

1) If $|\Omega| < \infty$, by [39, Theorem 2.1.5], the ultracontractivity implies that $\sigma(\Delta_R^p) = \sigma(\Delta_R)$.

2) If $|\Omega| = \infty$, then, by Theorem 4.2.11, the operator $e^{t\Delta_R}$ has a kernel K_t satisfying the upper bound

$$0 \leq K_t(x,y) \leq Ct^{-N} e^{\omega t} e^{-b\frac{|x-y|^2}{t}} \qquad (t > 0,\ x, y \in \Omega).$$

Thus the kernel $e^{-\omega t}K_t$ of $e^{t(\Delta_R - \omega I)}$ satisfies

$$0 \leq e^{-\omega t} K_t(x,y) \leq Ct^{-2N/2} e^{-b\frac{|x-y|^2}{t}} \qquad (t > 0,\ x, y \in \Omega).$$

Since $2N > N$, it follows from Theorem 4.2.13 that $\sigma(\Delta_R^p - \omega I) = \sigma(\Delta_R - \omega I)$ and therefore $\sigma(\Delta_R^p) = \sigma(\Delta_R)$ which completes the proof. $\square$

4.3 The Laplacian with Neumann Boundary Conditions.

This section is devoted to the study of the Laplacian with Neumann boundary conditions. We shall prove that $e^{t\Delta_N}$ is always an integral operator, but it is given by a singular kernel; i.e., a kernel which is not bounded if Ω is irregular.

Recall that as in Section 3.5, in the definition of Δ_N, if $\widetilde{H}^1(\Omega)$ is replaced by the closed linear subspace D_B of $\widetilde{H}^1(\Omega)$ of the form

$$D_B := \{u \in \widetilde{H}^1(\Omega) :\ \tilde{u} = 0 \ \text{ r.q.e. on } \ B\}$$

for some $B \in \mathcal{B}(\partial\Omega)$ with $\mathrm{Cap}_{\bar{\Omega}}(\partial\Omega \setminus B) > 0$, we call the selfadjoint operator Δ_N^B on $L^2(\Omega)$ associated to (a_N, D_B) the Laplacian with Dirichlet-Neumann boundary conditions.

A. Characterization of the Domain of Δ_N for Lipschitz Domains.

Similarly to the operator Δ_R, we can also give a characterization of the domain of Δ_N in the case where Ω has a Lipschitz boundary. Note that if Ω is irregular, a definition of a general notion of a "normal derivative" of a function u in $H(\Delta, \Omega)$ as before Proposition 4.2.7 for functions in $V(\Delta, \Omega)$ is not possible, because, we claim that, if the boundary of Ω is "bad" then, there is no $p \geq 1$ such that the space $\widetilde{H}^1(\Omega)/H_0^1(\Omega)$ is continuously embedded into $L^p(\partial\Omega, \sigma)$. Indeed, if for each bounded domain Ω with $\sigma(\partial\Omega) < \infty$, the space $\widetilde{H}^1(\Omega)/H_0^1(\Omega)$ is continuously embedded into $L^p(\partial\Omega, \sigma)$ for some $p \geq 1$, this shall imply that there exists a constant $C > 0$ such that

$$\sigma(K)^{2/p} \leq C\, \mathrm{Cap}_{\bar{\Omega}}(K)$$

for every compact set $K \subset \partial\Omega$ and then σ will be in $\mathcal{M}_0$. The domain of Example 2.3.10 says that this is not always the case.

Proposition 4.3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Then the operator Δ_N is given by*

$$\begin{cases} D(\Delta_N) & = \{u \in H(\Delta,\Omega) : \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\} \\ \Delta_N u & = \Delta u. \end{cases} \tag{4.38}$$

Proof. The proof is the same as for the operator Δ_R. For convenience, we shall repeat the proof. By definition

$$\begin{cases} D(\Delta_N) & := \{u \in H^1(\Omega) : \ \exists\, v \in L^2(\Omega) : \ a_N(u,\varphi) = (v,\varphi)\ \forall\, \varphi \in H^1(\Omega)\} \\ \Delta_N u & := -v. \end{cases}$$

Let

$$D := \{u \in H(\Delta,\Omega) : \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}.$$

Let $u \in D$ and $v := -\Delta u$. Then $v \in L^2(\Omega)$. Using the formula (4.22) we obtain that,

$$(v,\varphi) := (-\Delta u,\varphi) = a_N(u,\varphi) - \langle \gamma_1(u),\varphi\rangle \ \ \forall\, \varphi \in H^1(\Omega).$$

Since $u \in D$, it follows that $\langle \gamma_1(u),\varphi\rangle = 0$ for all $\varphi \in H^1(\Omega)$. Therefore $u \in D(\Delta_N)$ and $\Delta_N u = -v = \Delta u$.

To prove the converse inclusion, let $u \in D(\Delta_N)$. By hypothesis, there exists $v \in L^2(\Omega)$ such that

$$\int_\Omega \nabla u \nabla \varphi\, dx = \int_\Omega v\varphi\, dx \ \ \forall\, \varphi \in H^1(\Omega). \tag{4.39}$$

As for the operator Δ_R, the equality (4.39) implies that

$$-\Delta u = v \quad \text{in } \ \mathcal{D}(\Omega)'$$

and $\Delta u \in L^2(\Omega)$ and thus $u \in H(\Delta,\Omega)$. Using again the generalized Green formula (4.22), we obtain that $\langle \gamma_1(u),\varphi\rangle = 0$ for all $\varphi \in H^1(\Omega)$. Since $\varphi|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ is arbitrary, this implies that $\gamma_1(u) := \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$ in $H^{-1/2}(\partial\Omega)$ and thus $u \in D$ which completes the proof. $\square$

B. The Operator $e^{t\Delta_N}$ is always a Kernel Operator.

We introduce the following notion of *integral operators.* Let $1 \le p, q \le \infty$ and $\Omega \subset \mathbb{R}^N$ be an open set.

Definition 4.3.2. *a) A linear operator $T : L^p(\Omega) \to L^q(\Omega)$ is called an* **integral operator** *if there exists a measurable function $K : \Omega \times \Omega \to \mathbb{C}$ such that* *(i)* $K(x,\cdot)f(\cdot) \in L^1(\Omega)$ *x a.e. for all* $f \in L^p(\Omega)$, *and*

(ii) $(Tf)(x) = \int_\Omega K(x,y)f(y)\,dy$ *x a.e. for all* $f \in L^p(\Omega)$.

b) The operator T *is called* **regular** *if there exists a positive operator* $S \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ *such that* $|Tf| \le S|f|$ *for all* $f \in L^p(\Omega)$.

We denote by $\mathcal{L}^r(L^p(\Omega), L^q(\Omega))$ the space of all regular operators and by $I^r(L^p(\Omega), L^q(\Omega))$ the space of all **regular integral operators**. The following fundamental result is contained in [86, Chapter IV].

Theorem 4.3.3. *a) The space* $\mathcal{L}^r(L^p(\Omega), L^q(\Omega))$ *is a Banach lattice, the modulus* $|T|$ *of* $T \in \mathcal{L}^r(L^p(\Omega), L^q(\Omega))$ *is given by* $|T|f = \sup_{|g|\le f} |Tg|$ *and the norm by* $\|T\|_r := \|\,|T|\,\|$ *(the operator norm).*

b) $I^r(L^p(\Omega), L^q(\Omega))$ *is a closed subspace of* $\mathcal{L}^r(L^p(\Omega), L^q(\Omega))$.

c) $I^r(L^p(\Omega), L^q(\Omega))$ *is a band in* $\mathcal{L}^r(L^p(\Omega), L^q(\Omega))$.

Now we assume that $\Omega \subset \mathbb{R}^N$ is an arbitrary bounded open set.

Theorem 4.3.4. *Let* $T_N(t) := e^{t\Delta_N}$. *Then the following assertions hold.*

a) If $\sigma(\partial\Omega) < \infty$, *then* $T_N(t)$ *is an integral operator.*

b) If $\sigma(\partial\Omega) = \infty$, *then* $e^{t\Delta_N^{\Gamma_\infty}}$ *is an integral operator where* Γ_∞ *denotes the part of* $\partial\Omega$ *on which* σ *is locally infinite.*

Proof. We give a proof only for the case $\sigma(\partial\Omega) < \infty$. Let $T_{\sigma k} := (e^{t\Delta_{\sigma k}})_{t\ge 0}$ be the C_0-semigroup on $L^2(\Omega)$ generated by $\Delta_{\sigma k}$. We have proved in Theorem 4.2.11 that for each $k \ge 1$, $T_{\sigma k}(t) \in I^r(L^2(\Omega))$. Denote by $K_k(t,\cdot,\cdot)$ the kernel of $T_{\sigma k}(t)$. Then

$$(T_{\sigma k}(t)f)(x) = \int_\Omega K_k(t,x,y)f(y)\,dy$$

for x a.e. and all $f \in L^2(\Omega)$. Since $\sigma(\partial\Omega) < \infty$, by Proposition 3.5.5, $\Delta_{\sigma k} \to \Delta_N$ as $k \to \infty$ in the strong resolvent sense. Using the Second Trotter-Kato Approximation Theorem, one obtains that $T_{\sigma k}(t) \to T_N(t)$ as $k \to \infty$ strongly. Since $(T_{\sigma k}(t))$ is a directed family majorized by $T_N(t)$ (see Proposition 3.4.3); i.e.,

$$T_{\sigma 1}(t) \le T_{\sigma 2}(t) \le \dots T_{\sigma k}(t) \le \dots \le T_N(t)$$

one obtains that $T_N(t) = \sup_k T_{\sigma k}(t)$ and Theorem 4.3.3 c) completes the proof. □

Let K_t be the kernel of $T_N(t)$. Then

$$(T_N(t)f)(x) = \int_\Omega K_t(x,y)f(y)\,dy$$

for x a.e. and all $f \in L^2(\Omega)$. Note that K_t is not always in $L^\infty(\Omega \times \Omega)$. Indeed, assume that $0 \le K_t(x,y) \le a_t < \infty$ a.e. Then for $f \in L^1(\Omega)$ one has $\|T_N(t)f\|_\infty \le a_t\|f\|_1$; i.e.,

$$\|T_N(t)\|_{1\to\infty} \le a_t. \tag{4.40}$$

We obtain that

$$\|T_N(t)\|_{2\to\infty} \le a_t^{1/2}$$

by interpolating between the inequality (4.40) and the bound

$$\|T_N(t)\|_{\infty\to\infty} \le 1.$$

Then one has that $T_N(t)$ is ultracontractive. If $|\Omega| < \infty$, this implies that $T_N(t)$ is a compact semigroup on $L^2(\Omega)$. Since $(e^{t\Delta_N})_{t\ge 0}$ is norm continuous for $t > 0$, one obtains that Δ_N has a compact resolvent on $L^2(\Omega)$; i.e., the embedding $\widetilde{H}^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. The following example shows that this is not always the case.

Example 4.3.5. Let $\Omega \subset \mathbb{R}^2$ be the domain defined in Example 2.3.8 and $f \in C^\infty[0,1]$ be the function defined in the same example. Let (u_k) be the sequence of functions defined by

$$u_k(x,y) = \begin{cases} c_k f(y) & \text{if } a_k < x < b_k, \\ 0 & \text{otherwise} \end{cases}$$

where $c_k^2(b_k - a_k) = 1$. It is easy to see that $u_k \in H^1(\Omega) \cap C(\bar{\Omega})$. Moreover,

$$\begin{aligned} \|u_k\|_2^2 = \int_{A_k} c_k^2 |f(y)|^2\, dy &= (b_k - a_k)c_k^2 \int_{1/3}^1 |f(y)|^2\, dy \\ &= \int_{1/3}^1 |f(y)|^2\, dy. \end{aligned}$$

Similarly,

$$\|\nabla u_k\|_2^2 = \int_{1/3}^1 |f'(y)|^2\, dy.$$

Then (u_k) is a bounded sequence in $\widetilde{H}^1(\Omega)$. Furthermore, we have

$$\|u_k - u_l\|_2^2 = \int_{1/3}^1 |f(y)|^2\, dy > 0 \quad \text{if} \quad k \ne l.$$

Hence, there is no subsequence of (u_k) convergent in $L^2(\Omega)$. Thus, the embedding $\widetilde{H}^1(\Omega) \hookrightarrow L^2(\Omega)$ is noncompact.

C. Continuity and Compactness of the Embedding $\widetilde{H}^1(\Omega)$ into $L^p(\Omega)$.

In this subsection, we give some necessary and sufficient conditions on the relative capacity to have the compactness of the embedding $\widetilde{H}^1(\Omega) \hookrightarrow L^2(\Omega)$. We also give some sufficient conditions on the relative capacity to have a continuous embedding from $\widetilde{H}^1(\Omega)$ into $L^p(\Omega)$ for some $p > 2$.

Proposition 4.3.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For each compact set $K \subset \Omega$ we let*

$$\gamma(K) := \begin{cases} \frac{|K|}{\mathrm{Cap}_{\bar{\Omega}}(K)} & \text{if } \mathrm{Cap}_{\bar{\Omega}}(K) > 0 \\ 0 & \text{otherwise .} \end{cases}$$

The following assertions are equivalent.

(i) The embedding $\widetilde{H}^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

(ii) $\limsup_{\delta \to 0}\{\gamma(K) : \ K \subset \Omega, \ |K| \le \delta\} = 0$.

Proof. Recall that for each Borel set $B \subset \bar{\Omega}$ we have $|B| \le \mathrm{Cap}_{\bar{\Omega}}(B)$.
(i) $\Rightarrow$ (ii). Assume that (i) holds. Then for every $u \in H^1(\Omega) \cap C(\bar{\Omega})$, we have

$$\int_K |u|^2 \, dx \le \varepsilon(\delta) \|u\|^2_{H^1(\Omega)} \tag{4.41}$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$ and K is an arbitrary Borel subset of Ω with $|K| \le \delta$. Let K to be a compact set. If we insert into (4.41) an arbitrary function $u \in H^1(\Omega) \cap C(\bar{\Omega})$ with $u|_K \ge 1$ and pass to the infimum on the right part over such functions, one obtains

$$|K| \le \varepsilon(\delta) \, \mathrm{Cap}_{\bar{\Omega}}(K)$$

which gives (ii).

(ii) $\Rightarrow$ (i). Let $u \in H^1(\Omega) \cap C(\bar{\Omega})$ and let $\delta \in (0, \frac{|\Omega|}{2})$. Put

$$t(\delta) := \sup\{t \ge 0 : \ |E_t| \ge \delta\}$$

where $E_t := \{x \in \Omega : \ |u(x)| \ge t\}$. Then $t(\delta) < \infty$, $|E_{t(\delta)}| \ge \delta$ and $|E_t| < \delta$ for $t > t(\delta)$. We have

$$\begin{aligned} \int_\Omega |u|^2 \, dx &= \int_{E_{t(\delta)}} |u|^2 \, dx + \int_{E^c_{t(\delta)}} |u|^2 \, dx \\ &\le \int_{E_{t(\delta)}} |u|^2 \, dx + \int_\Omega (t(\delta))^2 \, dx \\ &\le \int_{E_{t(\delta)}} |u|^2 \, dx + |\Omega| t(\delta)^2. \end{aligned}$$

Let $v_\delta(x) := \max\{|u(x)| - t(\delta), 0\}$. Then

$$\|u\|_2 \le \|v_\delta\|_2 + t(\delta)|\Omega|^{1/2}. \tag{4.42}$$

Moreover,

$$\int_\Omega |v_\delta|^2\, dx = \int_0^\infty |\{x \in \Omega :\ v_\delta(x) \geq t\}|\ d(t^2).$$

Since $\{v_\delta \geq t\} \subset E_{t(\delta)+t}$ for all $t > 0$, it follows that

$$\int_0^\infty |\{v_\delta \geq t\}|\, d(t^2) \leq \int_0^\infty |E_{t(\delta)+t}|\, d(t^2).$$

We have $|E_{t(\delta)+t}| < \delta$ for $t > 0$ and by hypothesis

$$\begin{aligned}\int_0^\infty |E_{t(\delta)+t}|\, d(t^2) &\leq \int_0^\infty \sup \frac{|E_{t(\delta)+t}|}{\mathrm{Cap}_{\bar\Omega}(E_{t(\delta)+t})} \mathrm{Cap}_{\bar\Omega}(E_{t(\delta)+t})\, d(t^2) \\ &\leq f(\delta) \int_0^\infty \mathrm{Cap}_{\bar\Omega}(E_{t(\delta)+t})\, d(t^2)\end{aligned}$$

where $f(\delta)$ denotes the supremum in (ii). As $E_{t(\delta)+t} \subset E_t$, this implies that $\mathrm{Cap}_{\bar\Omega}(E_{t(\delta)+t}) \leq \mathrm{Cap}_{\bar\Omega}(E_t)$. By Proposition 2.2.5 we obtain

$$\|v_\delta\|_2 \leq cf(\delta)^{1/2} \left(\int_0^\infty \mathrm{Cap}_{\bar\Omega}(E_t)\, d(t^2) \right)^{1/2} \quad \leq \quad cf(\delta)^{1/2} \|u\|_{H^1(\Omega)}.$$

Moreover, since $|E_{t(\delta)}| \geq \delta$, we have that

$$\int_\Omega |u|\, dx \geq \int_{E_{t(\delta)}} |u|\, dx \geq \int_{E_{t(\delta)}} t(\delta)\, dx = t(\delta)|E_{t(\delta)}| \geq \delta t(\delta).$$

Finally, one has

$$\|u\|_2 \leq cg(\delta)\|u\|_{H^1(\Omega)} + \delta^{-1}|\Omega|^{1/2}\|u\|_1$$

where $c > 0$ and $g(\delta) \to 0$ as $\delta \to 0$. This inequality implies that every sequence in $H^1(\Omega) \cap C(\bar\Omega)$ bounded in $H^1(\Omega)$ and convergent in $L^1(\Omega)$, is convergent in $L^2(\Omega)$. Since the injection $\widetilde{H}^1(\Omega) \hookrightarrow L^1(\Omega)$ is compact, one obtains (i). □

Proposition 4.3.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $\sigma(\partial\Omega) < \infty$. Assume that there exists a constant $C > 0$ such that*

$$\sigma(K) \leq C\, \mathrm{Cap}_{\bar\Omega}(K) \tag{4.43}$$

for every compact set $K \subset \partial\Omega$. Then there exists a constant $M > 0$ such that

$$\|u\|_{\frac{2N}{N-1}} \leq M\|u\|_{H^1(\Omega)} \tag{4.44}$$

for all $u \in H^1(\Omega) \cap C(\bar\Omega)$.

Proof. Assume that the inequality (4.43) holds. By Proposition 2.2.6, this implies that

$$\|u\|_{L^2(\partial\Omega,\sigma)} \leq C\|u\|_{H^1(\Omega)} \tag{4.45}$$

for all $u \in H^1(\Omega) \cap C(\bar\Omega)$. Now replacing (4.45) in the Maz'ya inequality (4.12), we obtain (4.44). □

Theorem 4.3.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $\sigma(\partial\Omega) < \infty$. Then the following assertions are equivalent.*

(i) There exists a constant $C > 0$ such that

$$\sigma(K) \leq C \operatorname{Cap}_{\bar{\Omega}}(K) \tag{4.46}$$

for every compact set $K \subset \partial\Omega$.

(ii) There exists a constant $M > 0$ such that

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L^2(\partial\Omega,\sigma)} \leq M\|\nabla u\|_2 \tag{4.47}$$

for all $u \in H^1(\Omega) \cap C(\bar{\Omega})$.

Proof. (i) $\Rightarrow$ (ii). Assume that the inequality (4.46) holds. By Proposition 2.2.6, this implies that (4.45) holds and in particular that the embedding $\widetilde{H}^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Therefore the Poincaré inequality,

$$\inf_{c\in\mathbb{R}} \|u - c\|_2 \leq k\|\nabla u\|_2 \tag{4.48}$$

for all $u \in \widetilde{H}^1(\Omega)$, holds. Now replacing u by $u - c$ in (4.45), taking the infimum and using (4.48), we obtain

$$\begin{aligned} \inf_{c\in\mathbb{R}} \|u - c\|_{L^2(\partial\Omega,\sigma)} &\leq c_1\|\nabla u\|_2 + \inf_{c\in\mathbb{R}} \|u - c\|_2 \\ &\leq c_1\|\nabla u\|_2 + k\|\nabla u\|_2 \\ &\leq M\|\nabla u\|_2 \end{aligned}$$

which gives (ii).

(ii) $\Rightarrow$ (i). The inequality (4.47) implies that

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L^2(\partial\Omega,\sigma)} \leq M\|u\|_{H^1(\Omega)}. \tag{4.49}$$

for all $u \in H^1(\Omega) \cap C(\bar{\Omega})$. Let $K \subset \partial\Omega$ be a compact set and G a relatively open subset of $\partial\Omega$ such that $K \subset G$. Let

$$N := \{u \in H^1(\Omega) \cap C(\Omega) : \ u = 1 \ \text{ on } \ K, \ 0 \leq u \leq 1 \ \text{ and } \ u = 0 \ \text{ on } \ \partial\Omega \setminus G\}.$$

Substituting any function u of the class N into (4.49), we obtain

$$\min_{c\in\mathbb{R}} \left(|1 - c|^2\sigma(K) + c^2\sigma(\partial\Omega \setminus G)\right) \leq M^2 \operatorname{Cap}_{\bar{\Omega}}(K). \tag{4.50}$$

Computing, the minimum is attained for

$$c = \frac{\sigma(K)}{\sigma(K) + \sigma(\partial\Omega \setminus G)}.$$

Now, replacing c in (4.50), we obtain that

$$\frac{\sigma(K)\sigma(\partial\Omega\setminus G)}{\sigma(K)+\sigma(\partial\Omega\setminus G)} \leq M^2 \operatorname{Cap}_{\bar{\Omega}}(K).$$

If we assume that $2\sigma(G) \leq \sigma(\partial\Omega)$, then as $\sigma(\partial\Omega) = \sigma(G) + \sigma(\partial\Omega \setminus G)$ one has $\sigma(G) \leq \sigma(\partial\Omega \setminus G)$ and since $\sigma(K) \leq \sigma(G)$, this implies that $\sigma(K) + \sigma(\partial\Omega \setminus G) \leq 2\sigma(\partial\Omega \setminus G)$. Finally, we obtain

$$\frac{\sigma(K)\sigma(\partial\Omega\setminus G)}{2\sigma(\partial\Omega\setminus G)} \leq \frac{\sigma(K)\sigma(\partial\Omega\setminus G)}{\sigma(K)+\sigma(\partial\Omega\setminus G)} \leq M^2 \operatorname{Cap}_{\bar{\Omega}}(K)$$

which gives the inequality (4.46) with $C = 2M^2$. □

4.4 Comments.

Section 4.1.
All the results of this section are well-known and contained in [73] except inequality (4.11) which has been never considered by Maz'ya. Our definition of $W^1_{2,2}(\Omega, \partial\Omega)$ differs slightly from the Maz'ya one. In fact, for an arbitrary open set Ω, Maz'ya defines $W^1_{2,2}(\Omega, \partial\Omega)$ as the closure of W_σ with respect to the norm $|||\cdot|||_2$ given by (4.15). With this definition, if Ω has an infinite measure, then $W^1_{2,2}(\Omega, \partial\Omega)$ is not always a subspace of $L^2(\Omega)$ and in particular, it is not always embedded into $L^q(\Omega)$ for some $q > 2$. Our definition seemed to be the natural candidate for the study of forms for which subspaces of $L^2(\Omega)$ are needed.

The proof of Maz'ya's striking inequality (Theorem 4.1.7) given here is taken from [73, Theorem 3.6.3].

Section 4.2.
Properly speaking, Robin boundary conditions correspond to the case where $d\mu = \beta d\sigma$ for some positive measurable function β in $L^\infty(S, \sigma)$. Here, we consider the case $\beta = 1$, but all the results of this section are true if we take $\beta \in L^\infty(S, \sigma)$ with $\inf_S \beta(z) > 0$. Writing Δ_β for the operator associated with the closed form $(a_{\beta\sigma}, V)$, if $\beta = 0$ then $\Delta_0 = \Delta_N$ and if $\beta = \infty$ then $\Delta_\infty = \Delta_D$.

Assume that Ω is an open subset of $\mathbb{R}^N$ of class C^1. Let μ be a Borel measure on $\partial\Omega$ in $\mathcal{M}_0$. By [18, Proposition 5.2], if there exists $u \in D(\Delta_\mu) \cap C^2(\bar{\Omega})$ such that $u(z) > 0$ for all $z \in \partial\Omega$, then there exists $\beta \in C(\partial\Omega)_+$ such that $d\mu = \beta d\sigma$; i.e. $\Delta_\mu = \Delta_\beta$ and we have the classical Robin boundary conditions.

Conversely, assuming that Ω is of class $C^{2,\alpha}$ where $0 < \alpha < 1$, if $\beta \in C^{1,\alpha}(\partial\Omega)$ is such that $0 < \beta(z)$ $(z \in \partial\Omega)$, there always exists $u \in D(\Delta_\beta) \cap C^{2,\alpha}(\bar{\Omega})$ such that $\inf_{x\in\bar{\Omega}} u(x) > 0$ (see [18, Proposition 5.3]).

By the results of this section, given an open set Ω in $\mathbb{R}^N$, there (always) exists a natural subset S of $\partial\Omega$ where Robin boundary conditions are realized. First, we note that $\sigma(S) > 0$ if Ω is bounded. In fact, replacing u by the constant function

1 in the inequality (4.19) we obtain that $|\Omega|^{\frac{N-1}{N}} \leq C\sigma(S)$ and thus $\sigma(S) > 0$. Furthermore, we have shown that it is (always) possible to define a "normale derivative" on S in the generalized sense for every $u \in D(\Delta_R)$. We then see that the set S has some kind of regularity. Let

$$\mathcal{N} := \{x \in \partial\Omega : \text{ a normal to } \partial\Omega \text{ at } x \text{ exists}\}$$

be the **reduced boundary** of Ω (see [73, Section 6.2]). We don't know if $S = \mathcal{N}$.

To finish we note that Theorem 4.2.11 has been obtained by Daners [34] in the case where Ω is bounded.

Section 4.3.

The characterization of $D(\Delta_N)$ for bounded Lipschitz domains is well-known and contained in [38, Example 2 p.380].

Theorem 4.3.4 can also be obtained by using some results of Arendt and Bukhvalov [13]. In fact they have proved that for every bounded open set Ω in $\mathbb{R}^N$ the operator $R(\lambda, \Delta_N)$ (the resolvent of Δ_N) is always an integral operator for $\lambda > 0$. Since for each $u \in L^2(\Omega)$ the function $R(\lambda, \Delta_N)u$ is the Laplace transform of $e^{t\Delta_N}u$, Theorem 4.3.4 implies the result in [13]. Using the inverse Laplace transform, we can obtain the converse implication.

The proof of Proposition 4.3.6 is based on some ideas of Maz'ya and Poborchi [75, Theorem 8.6.3] where they give a similar equivalent condition to obtain the compactness of the embedding $L^1_p(\Omega)$ into $L^q(\Omega)$ for $1 < p \leq q < \infty$ where $L^1_p(\Omega)$ is the space of distributions on Ω with derivatives of order 1 in $L^p(\Omega)$.

Part II

Robin and Wentzell-Robin Laplacian on $C(\bar{\Omega})$

Chapter 5

The Robin Laplacian on $C(\bar{\Omega})$

Throughout this chapter, we assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with Lipschitz boundary. Recall that this means that $\partial\Omega$ is locally a graph of a Lipschitz function.

We shall denote by σ the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure which coincides with the usual Lebesgue surface measure (see Chapter 1).

Let $0 < \gamma \leq \beta \in L^\infty(\partial\Omega)$ and let $f \in L^p(\Omega),\ g \in L^q(\partial\Omega)$ where $p > N$ and $q \geq N$.

We consider the elliptic boundary value problem given formally by

$$\begin{cases} -\Delta u & = f \ \text{ in } \ \Omega; \\ \frac{\partial u}{\partial \nu} + \beta u & = g \ \text{ on } \partial\Omega \end{cases} \tag{5.1}$$

where ν denotes the exterior normal to $\partial\Omega$.

It is easy to see that this problem has a unique weak solution $u \in H^1(\Omega)$ (see Definition 5.1.1). The main result of this chapter (Theorem 5.2.7) says that $u \in C(\bar{\Omega})$. This is surprising since β is merely supposed to be measurable. We will use the method of De Giorgi applied by Murthy and Stampacchia [77] to solve elliptic equations with Dirichlet boundary conditions and Neumann boundary conditions if Ω is 1/2-admissible (see [77, Definition 9.7] for the definition). For the Dirichlet boundary conditions see also [12], [59] or [70]. For the case $\beta = 0$ and $g = 0$, i.e. the homogeneous Neumann boundary conditions, Fukushima and Tomisaki [58] prove that a weak solution is continuous on $\bar{\Omega}$. Biegert [22] gives an example of an open subset in $\mathbb{R}^2$ on which a weak solution of the homogenous Neumann problem (i.e., $\beta = 0$ and $g = 0$) is not continuous on $\bar{\Omega}$.

5.1 Preliminary Results.

Recall that Ω always denotes a bounded Lipschitz domain. Therefore the results of Chapter 4 imply that the Maz'ya space $W^1_{2,2}(\Omega, \partial\Omega)$ is isomorphic to $H^1(\Omega)$ with equivalent norm. In particular, this implies that $H^1(\Omega) \cap C(\bar{\Omega})$ is dense in $H^1(\Omega)$. Moreover, $H^1(\Omega)$ is continuously embedded into $L^{\frac{2N}{N-2}}(\Omega)$ and each function $u \in H^1(\Omega)$ has a trace noted $u|_{\partial\Omega}$ which is in $L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$; i.e. the trace application defined by

$$H^1(\Omega) \cap C(\bar{\Omega}) \longrightarrow L^{\frac{2(N-1)}{N-2}}(\partial\Omega), \qquad u \mapsto u|_{\partial\Omega} \tag{5.2}$$

has a continuous extension to $H^1(\Omega)$. Throughout the following, we let $s := \frac{2(N-1)}{N-2}$ and denote by s' the real number verifying $\frac{1}{s} + \frac{1}{s'} = 1$.

Definition 5.1.1. *A function $u \in H^1(\Omega)$ is called a weak solution of (5.1) if*

$$a_\sigma(u, v) = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, d\sigma, \qquad \forall\, v \in H^1(\Omega), \tag{5.3}$$

where we recall that for $u, v \in H^1(\Omega)$,

$$a_\sigma(u, v) := \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \beta u v \, d\sigma.$$

It is clear that the closed bilinear form a_σ is continuous on $H^1(\Omega)$ and it is **coercive** on $H^1(\Omega)$ in the sense that there exists a constant $c > 0$ such that for all $u \in H^1(\Omega)$

$$a_\sigma(u, u) \geq c\|u\|^2_{H^1(\Omega)}.$$

Let L be the linear functional on $H^1(\Omega)$ defined by: for $v \in H^1(\Omega)$ we let

$$Lv := \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, d\sigma.$$

Since $p \geq 2$ and $q \geq 2$, the functional L is well defined and continuous on $H^1(\Omega)$. Thus, by the coerciveness of the continuous bilinear form a_σ on $H^1(\Omega)$, the Lax-Milgram Lemma (see [24, Corollaire V.8 p.84]) implies that there exists a unique weak solution $u \in H^1(\Omega)$ of the boundary value problem (5.1).

For simplicity, all the calculations will be carried out assuming that $N \geq 3$. However, all the results hold also for $N = 2$ with minor changes.

Before starting this study, we recall some fundamental lemmas which we will use frequently.

Lemma 5.1.2. *Let $\varphi = \varphi(t)$ be a nonnegative, nonincreasing function on the half line $t \geq k_0 \geq 0$ such that there are positive constants c, α and δ ($\delta > 1$) such that*

$$\varphi(h) \leq c(h-k)^{-\alpha}\varphi(k)^\delta$$

for all $h > k \geq k_0$. Then we have:

$$\varphi(k_0 + d) = 0, \text{ where } d > 0 \text{ satisfies } d^\alpha = c\varphi(k_0)^{\delta-1}2^{\delta(\delta-1)}.$$

The proof of the following result given here is taken from [70, Theorem 2.106]. Note that Lemma 5.1.2 can be proved similarly as the following lemma by keeping r fixed in that case.

Lemma 5.1.3. *Let $\varphi = \varphi(t, r)$ be a nonnegative function on $(t \geq k_0 \geq 0) \times (0 \leq r < R_0)$ such that:*

a) for every fixed r, $\varphi(\cdot, r)$ is nonincreasing,

b) for every fixed h, $\varphi(h, \cdot)$ is nondecreasing

and such that there exist positive constants c, α, δ and γ ($\delta > 1$) with

$$\varphi(h, r) \leq c(h-k)^{-\alpha}(R-r)^{-\gamma}\varphi(k, R)^{\delta} \tag{5.4}$$

for all $h > k \geq k_0$, $r < R < R_0$. If σ is an arbitrary real number satisfying $0 < \sigma < 1$, then

$$\begin{cases} \varphi(k_0 + d, (1-\sigma)R_0) = 0, \\ \text{where } \ d^{\alpha} = c((1-\sigma)R_0)^{-\gamma}\varphi(k_0, R_0)^{\delta-1}2^{\delta\frac{\alpha+\gamma}{\delta-1}}. \end{cases}$$

Proof. Let $0 < \sigma < 1$, $k_j := k_0 + d - 2^{-j}d$, $r_j := (1-2^{-j})(1-\sigma)R_0$ and $\varphi_j := \varphi(k_j, r_j)$ where $j \in \mathbb{N}$. By (5.4), we have the following estimates:

$$\begin{array}{rcl} \varphi_j & \leq & c(2^{-j}d)^{-\alpha}(2^{-j}(1-\sigma)R_0)^{-\gamma}\varphi_{j-1}^{\delta} \\ & \leq & c2^{(\alpha+\gamma)j}d^{-\alpha}((1-\sigma)R_0)^{-\gamma}\varphi_{j-1}^{\delta} \\ & \leq & 2^{(\alpha+\gamma)(j-1-\frac{1}{\delta-1})}\varphi(k_0, R_0)^{1-\delta}\varphi_{j-1}^{\delta}. \end{array}$$

By induction we obtain that,

$$\varphi_j \leq 2^{-\frac{\alpha+\gamma}{\delta-1}j}\varphi(k_0, R_0).$$

Taking the limit as $j \to \infty$, we obtain the claim. □

The proof of the following result is contained in [77, Lemma 3.13].

Lemma 5.1.4. *Let φ be a nonnegative, nonincreasing function on a closed bounded interval $k_0 \leq t \leq M$ such that there exist positive constants c, α δ with*

$$(h-k)^{\alpha}\varphi(h)^{\delta} \leq c(M-k)^{\alpha}(\varphi(k) - \varphi(h)) \tag{5.5}$$

for all $k_0 \leq k < h < M$. Then $\lim_{h\to M}\varphi(h) = 0$.

The following well-known result is contained in [77, Lemma 11.3] (see also [59, Lemma 8.23 p.201] or [70, Lemma 4.12 p.196]).

Lemma 5.1.5. *Let $\omega(r)$ be a nonnegative function defined on $0 < \rho < R$ satisfying the condition that there exist constants η and c with $0 < \eta < 1$, $c \geq 0$ such that*

$$\omega(\rho) \leq \eta\omega(4\rho) + c\rho^{\alpha}, \quad 0 < 4\rho < R,\ \alpha > 0.$$

Then there exist two constants $K > 0$ and $0 < \delta < 1$ such that

$$\omega(r) \leq K r^{\delta} \quad \textit{for} \quad 0 < r < R.$$

Remark 5.1.6. *From the proof of the preceding lemma, we have that if $\alpha_0 > 0$ is a real number such that $4^{\alpha_0}\eta := c_1 < 1$ then $\delta = \min(\alpha, \alpha_0)$ and*

$$K = \left(\sup_{r \leq \rho \leq 4r} \frac{\omega(\rho)}{\rho^{\delta}} + \frac{c}{1 - c_1} \right).$$

Throughout the following, for $x_0 \in \bar{\Omega}$ and $r > 0$, we shall denote

$$\Omega(x_0, r) := \Omega \cap B(x_0, r) \quad \text{and} \quad \bar{\Omega}(x_0, r) := \bar{\Omega} \cap B(x_0, r),$$

where

$$B(x_0, r) := \{x \in \mathbb{R}^N : \ |x - x_0| < r\}.$$

If D is a Lebesgue measurable subset of Ω and E a σ-measurable subset of $\partial\Omega$, we denote by $|D|$ the Lebesgue measure of D and by $[E] := \sigma(E)$ the $(N-1)$-dimensional Hausdorff measure of E.

To obtain the uniform continuity on Ω of the solution of the problem (5.1), the following result will play an important role. It has been obtained by Fukushima and Tomisaki (see [58, Sections 3 and 6]).

Theorem 5.1.7. *Let $1 \leq p < N$ and r be a real number satisfying $p \leq r \leq \frac{pN}{N-p}$. Then for each $x_0 \in \bar{\Omega}$ and each $\kappa \in (0, 1]$, there exists a constant $\rho_0 := \rho_0(x_0) > 0$ such that,*

$$\left(\int_{\Omega(x_0,\rho)} |u|^r \, dx \right)^{1/r} \leq c(\kappa, r)\rho^{N(\frac{1}{r} - \frac{1}{p})} \left(\rho^p \int_{\Omega(x_0,\rho)} |\nabla u|^p \, dx + \int_D |u|^p \, dx \right)^{1/p} \tag{5.6}$$

for every $u \in H^1(\Omega(x_0, \rho))$ and every measurable subset D of $\Omega(x_0, \rho)$ such that $|D| \geq \kappa|\Omega(x_0, \rho)|$ and $0 < \rho \leq \rho_0$.

Remark 5.1.8. *a) For each $x_0 \in \bar{\Omega}$, we can always assume that $0 < \rho_0 < 1$. Then, as $pN(1/r - 1/p) + p \geq 0$, given a measurable subset D of $\Omega(x_0, \rho)$ such that $|D| \geq \kappa|\Omega(x_0, \rho)|$ for $0 < \rho < \rho_0$, the inequality (5.6) implies that*

$$\left(\int_{\Omega(x_0,\rho)} |u|^r \, dx \right)^{1/r} \leq c(\kappa, r) \left(\int_{\Omega(x_0,\rho)} |\nabla u|^p \, dx \right)^{1/p}$$

for every $u \in H^1(\Omega(x_0, \rho))$ such that $u = 0$ on D.

b) *The continuity of the trace application (5.2) gives that there exists a constant $C > 0$ depending only on Ω such that*

$$\|u\|_{s,\partial\Omega} \leq C\|u\|_{H^1(\Omega)}$$

for every $u \in H^1(\Omega)$. Then as in a), given a measurable subset D of $\Omega(x_0,\rho)$ such that $|D| \geq \kappa|\Omega(x_0,\rho)|$ for $0 < \rho < \rho_0$, we have that

$$\|u\|_{s,\partial\Omega\cap B(x_0,\rho)} \leq c(\kappa,\Omega)\|\nabla u\|_{2,\Omega(x_0,\rho)}$$

for every $u \in H^1(\Omega(x_0,\rho))$ such that $u = 0$ on D.

5.2 Hölder Continuity of Weak Solutions.

Throughout the following, if E is a subset of $\bar{\Omega}$, $u \in H^1(\Omega)$ and k a real number, we say that $u \geq k$ on E in the generalized sense, if there exists a sequence of functions $u_n \in C^1(\bar{\Omega})$ such that $u_n \geq k$ on E and u_n converges to u in $H^1(\Omega)$. Since a sequence of functions converging in $L^2(\Omega)$ has a subsequence converging almost everywhere with respect to the Lebesgue measure, it follows that $u \geq k$ on E in the generalized sense implies that $u \geq k$ on E a.e. Similarly we define $u \leq k$ on E in the generalized sense. A function $u \in H^1(\Omega)$ is said to be equal to k on E in the generalized sense when $u \geq k$ and $u \leq k$ on E in the generalized sense. If there is no confusion, we shall always omit the expression "generalized sense".

Before proving the Hölder continuity, we show that for some values of p and q the weak solution $u \in H^1(\Omega)$ of (5.1) is bounded on $\bar{\Omega}$.

Proposition 5.2.1. *Let $u \in H^1(\Omega)$ be a weak solution of (5.1) and assume that $p > N$ and $q > N-1$. Then there exists a constant $C = c(N,p,q,|\Omega|,[\partial\Omega]) > 0$ such that*

$$|u(x)| \leq C(\|f\|_{p,\Omega} + \|g\|_{q,\partial\Omega})$$

almost everywhere on $\bar{\Omega}$.

Proof. Let $u \in H^1(\Omega)$ be a weak solution of (5.1) and $k \geq 0$ be a real number. Let $v := (|u|-k)^+ \operatorname{sgn}(u)$. Then $v \in H^1(\Omega)$ is given by

$$v(x) = \begin{cases} u(x) - k & \text{if } u(x) \geq k \\ 0 & \text{if } |u(x)| \leq k \\ u(x) + k & \text{if } u(x) \leq -k. \end{cases}$$

Moreover,

$$\nabla v = \begin{cases} \nabla u & \text{in } A(k) \\ 0 & \text{otherwise} \end{cases}$$

where $A(k) := \{x \in \bar{\Omega} : \ |u(x)| > k\}$. Calculating, we obtain,

$$\begin{aligned} a_\sigma(u,v) &= \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \beta uv \, d\sigma \\ &= \int_{A(k)} |\nabla v|^2 \, dx + \int_{\partial\Omega \cap A(k)} \beta v^2 \, d\sigma + k \int_{\partial\Omega \cap A(k)} \beta |v| \, d\sigma \\ &= a_\sigma(v,v) + k \int_{\partial\Omega \cap A(k)} \beta |v| \, d\sigma \end{aligned}$$

which implies that

$$a_\sigma(v,v) \le a_\sigma(u,v) = \int_{A(k)} fv \, dx + \int_{A(k) \cap \partial\Omega} gv \, d\sigma. \tag{5.7}$$

Using the Hölder inequality, the continuous embedding from $H^1(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$ and the continuity of the trace application (5.2), we obtain the following estimates:

$$\begin{aligned} \int_{A(k)} fv \, dx &\le \|f\|_{2,A(k)} \|v\|_{2,A(k)} \\ &\le |A(k)|^{1/2-1/p} \|f\|_{p,\Omega} \|v\|_{H^1(\Omega)}. \end{aligned} \tag{5.8}$$

Similarly,

$$\begin{aligned} \int_{A(k)\cap\partial\Omega} gv \, d\sigma &\le \|g\|_{2,A(k)\cap\partial\Omega} \|v\|_{2,A(k)\cap\partial\Omega} \\ &\le [A(k) \cap \partial\Omega]^{1-1/q-1/s} \|g\|_{q,\partial\Omega} \|v\|_{H^1(\Omega)}. \end{aligned} \tag{5.9}$$

Letting

$$H(k) := |A(k)|^{1/2-1/p} \|f\|_{p,\Omega} + [A(k) \cap \partial\Omega]^{1-1/s-1/q} \|g\|_{q,\partial\Omega},$$

the inequalities (5.7), (5.8) and (5.9) imply that

$$\|v\|_{H^1(\Omega)} \le cH(k).$$

It follows that there exist some constants $c_1, c_2 > 0$ such that

$$\|v\|_{\frac{2N}{N-2},A(k)} \le c_1 H(k) \tag{5.10}$$

and

$$\|v\|_{s,A(k)\cap\partial\Omega} \le c_2 H(k). \tag{5.11}$$

Let $h > k \ge 0$. Then $A(h) \subset A(k)$ and on $A(h)$ we have $|v| \ge h - k$. Thus the inequalities (5.10) and (5.11) imply that

$$|A(h)|^{\frac{N-2}{N}} \le c_1 (h-k)^{-2} H(k)^2 \tag{5.12}$$

and

$$[A(k)\cap\partial\Omega]^{\frac{2}{s}} \leq c_2(h-k)^{-2}H(k)^2. \tag{5.13}$$

Next letting $a(h) := |A(h)| + [A(h)\cap\partial\Omega]^{\frac{2N}{s(N-2)}}$, we obtain the following estimates:

$$\begin{aligned} a(h) &\leq (h-k)^{-2^*}H(k)^{2^*} \\ &\leq c(h-k)^{-2^*}\left(|A(k)|^{1/2-1/p}\|f\|_{p,\Omega} + [A(k)\cap\partial\Omega]^{1-1/s-1/q}\|g\|_{q,\partial\Omega}\right)^{2^*} \\ &\leq c(h-k)^{-2^*}\left(a(k)^{1/2-1/p}\|f\|_{p,\Omega} + a(k)^{(1-1/s-1/q)2^*}\|g\|_{q,\partial\Omega}\right)^{2^*}, \end{aligned}$$

where $2^* := \frac{2N}{N-2}$. Let

$$\delta := \min\{(1/2-1/p)2^*, (1-1/q-1/s)s\}. \tag{5.14}$$

Then

$$\begin{aligned} a(h) &\leq c(h-k)^{-2^*}\Big(a(k)^{((1/2-1/p)2^*-\delta)\frac{1}{2^*}}\|f\|_{p,\Omega} + \\ &\qquad +a(k)^{((1-1/s-1/q)s-\delta)\frac{1}{2^*}}\|g\|_{q,\partial\Omega}\Big)^{2^*}a(k)^\delta \\ &\leq c(h-k)^{-2^*}\Big(|\Omega|^{((1/2-1/p)2^*-\delta)\frac{1}{2^*}}\|f\|_{p,\Omega} + \\ &\qquad +[\partial\Omega]^{((1-1/s-1/q)s-\delta)\frac{1}{2^*}}\|g\|_{q,\partial\Omega}\Big)^{2^*}a(k)^\delta. \end{aligned}$$

Let

$$c_3 := c\max\{|\Omega|^{(1/2-1/p)2^*-\delta}, [\partial\Omega]^{(1-1/q-1/s)s-\delta}\}.$$

We finally obtain that

$$a(h) \leq c_3(h-k)^{-2^*}\left(\|f\|_{p,\Omega} + \|g\|_{q,\partial\Omega}\right)^{2^*}a(k)^\delta.$$

Since $p > N$ and $q > N-1$, it follows that $\delta > 1$. The function $a(h)$ satisfies then the conditions of Lemma 5.1.2 with $\alpha = 2^* = \frac{2N}{N-2}$ and δ given by (5.14). Taking $k_0 = 0$, one obtains that $a(d) = 0$ with

$$d^\alpha \leq c_3\left(\|f\|_{p,\Omega} + \|g\|_{q,\partial\Omega}\right)^{2^*}a(0)^{\delta-1} = c(\|f\|_{p,\Omega} + \|g\|_{q,\partial\Omega})^\alpha$$

which implies that

$$|u(x)| \leq C(\|f\|_{p,\Omega} + \|g\|_{q,\partial\Omega})$$

almost everywhere on $\bar{\Omega}$. □

For $g = 0$, the result of the preceding proposition has been obtained by Daners (see [34, Theorem 4.2]) with a different proof and for an arbitrary bounded domain Ω.

We obtain the following result as a corollary of the proof of the preceding proposition.

Corollary 5.2.2. *Let $f_i \in L^p(\Omega), i = 0, \ldots, N,\ g \in L^q(\Omega)$ and $u \in H^1(\Omega)$ be such that*

$$a_\sigma(u, \varphi) = \int_\Omega f_0 \varphi \, dx + \sum_{i=1}^{N} \int_\Omega f_i D_i \varphi \, dx + \int_{\partial\Omega} g\varphi \, d\sigma$$

for all $\varphi \in H^1(\Omega)$. Assume that $p > N$ and $q > N-1$. Then there exists a constant $C = c(N, p, q, |\Omega|, [\partial\Omega]) > 0$ such that

$$|u(x)| \leq C(\sum_{i=0}^{N} \|f_i\|_{p,\Omega} + \|g\|_{q,\partial\Omega})$$

almost everywhere on $\bar{\Omega}$.

Before giving the main result, we need some preparations. Recall that Ω always denotes a bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geq 3)$.

Proposition 5.2.3. *There exists a constant $C > 0$ such that for each $x_0 \in \bar{\Omega}$, and for all $k \geq 0$ and for all $0 < \rho < R < \rho_0$, a weak solution $u \in H^1(\Omega)$ of the equation (5.1) satisfies*

$$\int_{A(k,\rho)} |\nabla u|^2 \, dx \leq C \left\{ (R-\rho)^{-2} \int_{A(k,R)} |u-k|^2 \, dx + G(k, R) \right\} \tag{5.15}$$

where for $0 < r < \rho_0$,

$$A(k, r) := \{x \in \bar{\Omega}(x_0, r) : \ u(x) > k\},$$

and

$$G(k, R) := \|f\|^2_{p,\partial\Omega} |A(k, R)|^{1-2/p+2/N} + \|g\|^2_{q,\partial\Omega} [A(k, R) \cap \partial\Omega]^{2/s'-2/q}.$$

Proof. Let $x_0 \in \bar{\Omega}$ be fixed and $\rho_0 > 0$ be the constant given by Theorem 5.1.7. Let ρ and R be two real numbers verifying $0 < \rho < R < \rho_0$ and let $\psi \in C^1_c(B(x_0, \rho_0))$ be a function such that

$$\begin{cases} 0 \leq \psi(x) \leq 1 \\ |\nabla\psi(x)| \leq c(R-\rho)^{-1} \end{cases} \quad \psi(x) = \begin{cases} 1 & \text{in } \ B(x_0, \rho) \\ 0 & \text{outside } \ B(x_0, R). \end{cases}$$

Let $u \in H^1(\Omega)$ be a weak solution of (5.1) and k be a real number. We consider the function $v \in H^1(\Omega)$ defined by $v := \psi^2(u-k)^+$. Then

$$v = \begin{cases} \psi^2(u-k) & \text{in } \ A(k, R) \\ 0 & \text{outside } . \end{cases}$$

Replacing v in (5.3) we obtain:

$$\int_{A(k,R)} \nabla u \nabla[\psi^2(u-k)]\,dx + \int_{A(k,R)\cap\partial\Omega} \beta u \psi^2(u-k)\,d\sigma =$$
$$\int_{A(k,R)} f\psi^2(u-k)\,dx + \int_{A(k,R)\cap\partial\Omega} g\psi^2(u-k)\,d\sigma.$$

Since $u(u-k) \geq 0$ on $A(k,R)$,

$$\begin{aligned}
\int_{A(k,R)} \psi^2|\nabla u|^2\,dx \;\leq\; & -2\int_{A(k,R)} \psi\nabla u\nabla\psi(u-k)\,dx \\
& + \int_{A(k,R)} \psi f\psi(u-k)\,dx + \int_{A(k,R)\cap\partial\Omega} \psi g\psi(u-k)\,d\sigma.
\end{aligned}$$

Next, using the Hölder inequality, the Young inequality, the continuity of the trace application (5.2) and the continuous embedding from $H^1(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$, we obtain the following estimates:

$$\begin{aligned}
-2\int_{A(k,R)} \psi\nabla u\nabla\psi(u-k)\,dx &\leq 2\|\psi\nabla u\|_{2,A(k,R)}\|\nabla\psi(u-k)\|_{2,A(k,R)} \\
&\leq \varepsilon\|\psi\nabla u\|^2_{2,A(k,R)} + \frac{1}{\varepsilon}\|\nabla\psi(u-k)\|^2_{2,A(k,R)}
\end{aligned}$$

for every $\varepsilon > 0$. Similarly,

$$\begin{aligned}
\int_{A(k,R)} \psi f\psi(u-k)\,dx &\leq \|\psi f\|_{2,A(k,R)}\|\psi(u-k)\|_{2,A(k,R)} \\
&\leq \|\psi f\|_{2,A(k,R)}|A(k,R)|^{1/N}\|\psi(u-k)\|_{\frac{2N}{N-2},A(k,R)} \\
&\leq c_1\|\psi f\|_{2,A(k,R)}|A(k,R)|^{1/N}\|\psi(u-k)\|_{H^1(\Omega)} \\
&\leq \varepsilon c_1\|\psi(u-k)\|^2_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\psi f\|^2_{2,A(k,R)}|A(k,R)|^{2/N} \\
&\leq \varepsilon c_1\|\psi\nabla u\|^2_{2,A(k,R)} + \varepsilon c_1\|\psi(u-k)\|^2_{2,A(k,R)} \\
&+ c_1\varepsilon\|\nabla\psi(u-k)\|^2_{2,A(k,R)} + \frac{1}{\varepsilon}\|\psi f\|^2_p|A(k,R)|^{1-2/p+2/N}
\end{aligned}$$

for every $\varepsilon > 0$ and some constant $c_1 = c(p,q,|\Omega|,[\partial\Omega]) > 0$. Moreover,

$$\begin{aligned}
\int_{A(k,R)\cap\partial\Omega} \psi g\psi(u-k)\,d\sigma &\leq \|\psi g\|_{s',A(k,R)\cap\partial\Omega}\|\psi(u-k)\|_{s,A(k,R)\cap\partial\Omega} \\
&\leq \frac{1}{2\varepsilon}|\psi g\|^2_{s',A(k,R)\cap\partial\Omega} + c_2\varepsilon\|\psi(u-k)\|^2_{H^1(\Omega)} \\
&\leq \varepsilon c_2\|\psi\nabla u\|^2_{2,A(k,R)} + \varepsilon c_2\|\psi(u-k)\|^2_{2,A(k,R)} \\
&\quad + \varepsilon c_2\|\nabla\psi(u-k)\|^2_{2,A(k,R)} \\
&\quad + \frac{1}{2\varepsilon}\|\psi g\|^2_{q,\partial\Omega}[A(k,R)\cap\partial\Omega]^{2/s'-2/q}
\end{aligned}$$

for every $\varepsilon > 0$ and some constant $c_2 = c(p, q, |\Omega|, [\partial\Omega]) > 0$. Choosing ε suitably, using the fact that $\psi = 1$ on $B(x_0, \rho)$, $0 \le \psi(x) \le 1$, $|\nabla\psi(x)| \le c(R - \rho)^{-1}$ and $A(k, \rho) \subset A(k, R)$, we obtain the inequality (5.15). $\square$

Lemma 5.2.4. *There exist some constants $C_1, C_2, C_3 > 0$ such that for each $x_0 \in \bar{\Omega}$, and for all $0 < \rho < \rho_0$ and for every $u \in H^1(\Omega)$, we have*

(i) $\int_{A(k,\rho)} |u - k|^2\, dx \le C_1 |A(k, \rho)|^{2/N} \int_{A(k,\rho)} |\nabla u|^2\, dx$;

(ii) $|A(h, \rho)|^{\frac{N-2}{N}} \le C_2 (h - k)^{-2} \int_{A(k,\rho)\setminus A(h,\rho)} |\nabla u|^2\, dx$;

(iii) $|A(h, \rho)|^{\frac{2N-2}{N}} \le C_3 (h - k)^{-2} \int_{A(k,\rho)} |\nabla u|^2\, dx\, (|A(k, \rho)| - |A(h, \rho)|)$

for all $h > k \ge 0$ and for every $k \ge 0$ such that

$$|A(k, \rho)| \le \frac{1}{2} |\Omega(x_0, \rho)|,$$

where $A(k, \rho)$ is defined as in Proposition 5.2.3.

Proof. Let $x_0 \in \bar{\Omega}$, $0 < \rho < \rho_0$ and $k \ge 0$ be a real number such that $|A(k, \rho)| \le \frac{1}{2} |\Omega(x_0, \rho)|$ and let $u \in H^1(\Omega)$.

(i) Consider the function $v \in H^1(\Omega)$ defined by $v := (u - k)^+$. Then

$$\int_\Omega |v|^2\, dx = \int_{A(k,\rho)} |u - k|^2\, dx \le |A(k, \rho)|^{2/N} \left(\int_{A(k,\rho)} |u - k|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}.$$

Since $v = 0$ on $\Omega(x_0, \rho) \setminus A(k, \rho)$ and $|\Omega(x_0, \rho) \setminus A(k, \rho)| > \frac{1}{2} |\Omega(x_0, \rho)|$, by Theorem 5.1.7, there exists a constant $C_1 > 0$ such that

$$\left(\int_{A(k,\rho)} |u - k|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \le C_1 \int_{A(k,\rho)} |\nabla u|^2\, dx.$$

Finally, this implies that

$$\int_{A(k,\rho)} |u - k|^2\, dx \le C_1 |A(k, \rho)|^{2/N} \int_{A(k,\rho)} |\nabla u|^2\, dx.$$

(ii) Let $h > k$, where $k \ge 0$ satisfies $|A(k, \rho)| \le \frac{1}{2} |\Omega(x_0, \rho)|$. Consider the functions $v_1, v_2 \in H^1(\Omega)$ defined by $v_1 = (u - k)^+$, $v_2 = (u - h)^+$ and let $w := v_1 - v_2$. Then

$$w = \begin{cases} h - k & \text{in } A(h, \rho) \\ u - k & \text{in } A(k, \rho) \setminus A(h, \rho) \\ 0 & \text{otherwise .} \end{cases}$$

Since $w = 0$ on $\Omega(x_0,\rho) \setminus A(k,\rho)$ and $|\Omega(x_0,\rho) \setminus A(k,\rho)| > \frac{1}{2}|\Omega(x_0,\rho)|$, using the Hölder inequality and Theorem 5.1.7, we obtain the following estimates:

$$\begin{aligned}
\int_{A(h,\rho)} |w|^2\,dx &\leq |A(h,\rho)|^{2/N}\left(\int_{A(h,\rho)} |w|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \\
&\leq |A(h,\rho)|^{2/N}\left(\int_{A(k,\rho)} |w|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \\
&\leq C_2|A(h,\rho)|^{2/N}\int_{A(k,\rho)} |\nabla w|^2\,dx.
\end{aligned}$$

for some constant $C_2 > 0$. Replacing w, we obtain that

$$|A(h,\rho)| \leq C_2(h-k)^{-2}|A(h,\rho)|^{2/N}\int_{A(k,\rho)} |\nabla u|^2\,dx$$

which gives (ii).

(iii) Consider the same function $w \in H^1(\Omega)$ as in (ii). Since $w = 0$ on $\Omega(x_0,\rho) \setminus A(k,\rho)$ and $|\Omega(x_0,\rho) \setminus A(k,\rho)| > \frac{1}{2}|\Omega(x_0,\rho)|$, using the Hölder inequality and Theorem 5.1.7, we obtain the following estimates:

$$\begin{aligned}
\int_{A(h,\rho)} |w|\,dx &\leq |A(h,\rho)|^{1/N}\left(\int_{A(h,\rho)} |w|^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}} \\
&\leq |A(h,\rho)|^{1/N}\left(\int_{A(k,\rho)} |w|^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}} \\
&\leq c|A(h,\rho)|^{1/N}\int_{A(k,\rho)} |\nabla w|\,dx
\end{aligned}$$

for some constant $c > 0$. Replacing w, we obtain the following estimates:

$$\begin{aligned}
|A(h,\rho)| &\leq c(h-k)^{-1}|A(h,\rho)|^{1/N}\int_{A(k,\rho)\setminus A(h,\rho)} |\nabla u|\,dx \\
&\leq c(h-k)^{-1}|A(h,\rho)|^{\frac{1}{N}}|A(k,\rho) \setminus A(h,\rho)|^{\frac{1}{2}} \times \\
&\qquad \times \left(\int_{A(k,\rho)\setminus A(h,\rho)} |\nabla u|^2\,dx\right)^{\frac{1}{2}}
\end{aligned}$$

which implies that

$$|A(h,\rho)|^{\frac{2N-2}{N}} \leq C_3(h-k)^{-2}\int_{A(k,\rho)} |\nabla u|^2\,dx\,(|A(k,\rho)| - |A(h,\rho)|)\,.$$

□

Next we give some estimates for a weak solution u of (5.1).

Proposition 5.2.5. *For each $x_0 \in \bar{\Omega}$, and for all $0 < \rho < R < \rho_0$ and for a weak solution $u \in H^1(\Omega)$ of (5.1) the following estimates hold.*

(i) $\int_{A(h,\rho)} |u-h|^2\,dx \le c_1 \left\{(R-\rho)^{-2}\int_{A(k,R)} |u-k|^2\,dx + G(k,R)\right\} \times |A(k,R)|^{2/N}$;

(ii) $|A(h,\rho)|^{\frac{N-2}{N}} \le c_2 (h-k)^{-2}\left\{(R-\rho)^{-2}\int_{A(k,R)} |u-k|^2\,dx + G(k,R)\right\}$;

(iii) $|A(h,\rho)|^{\frac{2N-2}{N}} \le c_3 (h-k)^{-2}\left\{(R-\rho)^{-2}\int_{A(k,R)} |u-k|^2\,dx + G(k,R)\right\} \times (|A(k,\rho)| - |A(h,\rho)|)$;

(iv) $[A(h,\rho)\cap\partial\Omega]^{\frac{2}{s}} \le c_4 (h-k)^{-2}\left\{(R-\rho)^{-2}\int_{A(k,R)} |u-k|^2\,dx + G(k,R)\right\}$

for all $k \ge 0$ such that $|A(k,\rho)| \le \frac{1}{2}|\Omega(x_0,\rho)|$ and for all $h > k$, where $G(k,R)$ and $A(k,\rho)$ are defined as in Proposition 5.2.3.

Proof. (i) Since $A(h,\rho) \subset A(k,\rho)$, by Lemma 5.2.4 (i),

$$\int_{A(h,\rho)} |u-h|^2\,dx \le \int_{A(k,\rho)} |u-k|^2\,dx \le c|A(k,\rho)|^{2/N}\int_{A(k,\rho)} |\nabla u|^2\,dx.$$

Now we obtain (i) by using Proposition 5.2.3.

The assertions (ii) and (iii) are also an easy consequence of Lemma 5.2.4 and Proposition 5.2.3.

(iv) Let $w \in H^1(\Omega)$ be the function defined in the proof of Lemma 5.2.4 (ii). Since $w = 0$ on $\Omega(x_0,\rho)\setminus A(k,\rho)$ and $|\Omega(x_0,\rho)\setminus A(k,\rho)| > \frac{1}{2}|\Omega(x_0,\rho)|$ we obtain

$$\left(\int_{A(h,\rho)\cap\partial\Omega} |w|^s\,d\sigma\right)^{2/s} \le \left(\int_{A(k,\rho)\cap\partial\Omega} |w|^s\,d\sigma\right)^{2/s} \le c\int_{A(k,\rho)} |\nabla w|^2\,dx.$$

Replacing w, we obtain that

$$[A(h,\rho)\cap\partial\Omega]^{2/s} \le c(h-k)^{-2}\int_{A(k,\rho)} |\nabla u|^2\,dx.$$

Now Proposition 5.2.3 completes the proof. $\square$

Next we introduce some notations. Let $E \subset \Omega$ be an open set. We put

$$H(E) := \{u|_E : \ u \in H^1(\Omega)\}.$$

Let G be an open set, not necessarily contained in Ω. We denote by $H^1_c(G)$ the closure in $H^1(\Omega)$ of the space of functions of the type ψu where $u \in H^1(\Omega)$ and $\psi \in C^1_c(\mathbb{R}^N)$ with $\operatorname{supp}\psi \subset G$.

Remark 5.2.6. *If $G \subset \bar{G} \subset \Omega$, then $H_c^1(G) = H_0^1(G)$. Note that in order to obtain the estimates of Proposition 5.2.3 at a fixed point $x_0 \in \bar{\Omega}$, it is enough that $u \in H(\Omega(x_0, R))$ and (5.3) is satisfied by those functions $\psi \in H_c^1(B(x_0, R))$.*

We are now in position to prove our main result which is the following theorem.

Theorem 5.2.7. *Let $p > N$ and $q > N - 1$. Then there exists a constant $0 < \delta < 1$ such that for every $f \in L^p(\Omega)$ and $g \in L^q(\partial\Omega)$ the weak solution $u \in H^1(\Omega)$ of (5.1) belongs to $C^{0,\delta}(\bar{\Omega})$.*

Proof. Let $p > N$ and $q > N - 1$. We show that there exists a constant $0 < \delta_1 < 1$ such that if $f \in L^p(\Omega)$, $g \in L^q(\partial\Omega)$ and $u \in H^1(\Omega)$ is the weak solution of (5.1), then there exist a constant $H > 0$ such that

$$\omega(r) \leq H r^{\delta_1}, \tag{5.16}$$

for every $x_0 \in \bar{\Omega}$ and for every $0 < r < \rho_0$, where

$$\omega(r) := \operatorname*{ess\,sup}_{\bar{\Omega}(x_0,r)} u(x) - \operatorname*{ess\,inf}_{\bar{\Omega}(x_0,r)} u(x).$$

By Remark 5.2.6, it is enough to consider a solution $u \in H(\Omega(x_0, R))$ of the equation

$$\int_{\Omega(x_0,R)} \nabla u \nabla \varphi + \int_{\bar{\Omega}(x_0,R)\cap\partial\Omega} \beta u \varphi \, d\sigma = \int_{\Omega(x_0,R)} f\varphi \, dx + \int_{\bar{\Omega}(x_0,R)\cap\partial\Omega} g\varphi \, d\sigma \tag{5.17}$$

for all $\varphi \in H_c^1(B(x_0, R))$. By definition, for every open set G, $H_c^1(G)$ is a closed subspace of $H^1(\Omega)$. Since a_σ is a closed coercive form on $H^1(\Omega)$, it follows that a_σ is a closed coercive form on $H_c^1(G)$. Let $G := B(x_0, R)$ and L be the functional defined by: for $\varphi \in H_c^1(B(x_0, R))$ we let

$$L\varphi := \int_{\Omega(x_0,R)} f\varphi \, dx + \int_{\bar{\Omega}(x_0,R)\cap\partial\Omega} g\varphi \, d\sigma.$$

Then L is a linear continuous functional on $H_c^1(B(x_0, R))$. Since a_σ is a closed coercive form on $H_c^1(B(x_0, R))$, it follows from the Lax-Milgram Lemma that the equation (5.17) restricted to $H_c^1(B(x_0, R))$ has a unique solution $w \in H_c^1(B(x_0, R))$.

Now, let $u \in H(\Omega(x_0, R))$ be a solution of (5.17) and $v := u - w$. Then $v \in H(\Omega(x_0, R))$ satisfies

$$\int_{\Omega(x_0,R)} \nabla v \nabla \varphi + \int_{\bar{\Omega}(x_0,R)\cap\partial\Omega} \beta v \varphi \, d\sigma = 0 \qquad \forall\, \varphi \in H_c^1(B(x_0, R)). \tag{5.18}$$

Thus, if $u \in H(\Omega(x_0, R))$ is a solution of (5.17), we can always decompose $u = v + w$, where $v \in H(\Omega(x_0, R))$ is a solution of (5.18) and $w \in H_c^1(B(x_0, R))$ satisfies

$$\int_{\Omega(x_0,R)} \nabla w \nabla \varphi + \int_{\bar{\Omega}(x_0,R)\cap\partial\Omega} \beta w \varphi \, d\sigma = \int_{\Omega(x_0,R)} f\varphi \, dx + \int_{\bar{\Omega}(x_0,R)\cap\partial\Omega} g\varphi \, d\sigma, \tag{5.19}$$

for all $\varphi \in H^1_c(B(x_0,R))$.

1) We claim that there exists a constant $0 < \eta < 1$ such that for $v \in H(\Omega(x_0,4r))$ solution of (5.18), we have

$$\omega_1(r) \le \eta\omega_1(4r) \tag{5.20}$$

where for $0 < \rho < \rho_0$,

$$\omega_1(\rho) := \operatorname*{ess\,sup}_{\bar{\Omega}(x_0,\rho)} v(x) - \operatorname*{ess\,inf}_{\bar{\Omega}(x_0,\rho)} v(x).$$

Indeed, let $k_0 \ge 0$ and $k \ge k_0$ be such that $|A(k,\rho)| \le \frac{1}{2}|\Omega(x_0,\rho)|$ for every $0 < \rho < \rho_0$, where here

$$A(k,\rho) := \{x \in \bar{\Omega}(x_0,\rho) :\ v(x) > k\}.$$

Let $h > k \ge k_0$, and $0 < \rho < R < \rho_0$ and set

$$\mu(h,\rho) := \int_{A(h,\rho)} |v-h|^2\, dx$$

and

$$a(h,\rho) := |A(h,\rho)| + [A(h,\rho)\cap\partial\Omega]^{\frac{2N}{s(N-2)}}.$$

Since v satisfies the estimates of Propositions 5.2.3 and 5.2.5 with $G(k,R) = 0$, by Proposition 5.2.5 (i) one has,

$$\begin{aligned} \mu(h,\rho) &\le c(R-\rho)^{-2}\mu(k,R)|A(k,R)|^{2/N} \\ &\le c(R-\rho)^{-2}\mu(k,R)a(k,R)^{2/N}. \end{aligned}$$

Let α be the positive solution of the equation

$$2\alpha^2 = (\alpha+1)(N-2). \tag{5.21}$$

Then,

$$\mu(h,\rho)^\alpha \le c(R-\rho)^{-2\alpha}\mu(k,R)^\alpha a(k,R)^{\frac{2\alpha}{N}}. \tag{5.22}$$

By Proposition 5.2.5 (ii) and (iv),

$$a(h,\rho)^{\frac{N-2}{N}} \le c(h-k)^{-2}(R-\rho)^{-2}\mu(k,R). \tag{5.23}$$

The inequalities (5.22) and (5.23) imply that

$$\mu(h,\rho)^\alpha a(h,\rho)^{\frac{N-2}{N}} \le c(h-k)^{-2}(R-\rho)^{-2-2\alpha}\mu(k,R)^{1+\alpha}a(k,R)^{\frac{2\alpha}{N}}.$$

Letting $\varphi(h,\rho) = a(h,\rho)^{\frac{N-2}{N}}\mu(h,\rho)^\alpha$, we obtain

$$\varphi(h,\rho) \le c(h-k)^{-2}(R-\rho)^{-2-2\alpha}\varphi(k,R)^\delta$$

where $\delta = 1 + \frac{1}{\alpha} > 1$. Then the function $\varphi(h, \rho)$ satisfies the conditions of Lemma 5.1.3. Taking $R = 2r$ and $\rho = r$ we obtain that $\varphi(k_0 + d, r) = 0$ where

$$
\begin{aligned}
d^2 &= cr^{-2-2\alpha}\varphi(k_0, 2r)^{\delta-1} \\
&\leq C\, r^{-2-2\alpha} r^{\frac{N-2}{\alpha}} \mu(k_0, 2r) \\
&= C\, r^{-N} \mu(k_0, 2r).
\end{aligned}
$$

But $\varphi(k_0 + d, r) = 0$ implies that $|A(k_0 + d, r)| = [A(k_0 + d, r) \cap \partial\Omega] = 0$ which implies that $v(x) \leq k_0 + d$ almost everywhere on $\bar{\Omega}(x_0, r)$. Then we obtain that

$$
\operatorname*{ess\,sup}_{\bar{\Omega}(x_0,r)} v(x) \leq k_0 + \left(Cr^{-N} \int_{A(k_0,2r)} |v - k_0|^2 \, dx \right)^{1/2}. \tag{5.24}
$$

Next for $0 < \rho < \rho_0$ we put

$$
M_1(\rho) := \operatorname*{ess\,sup}_{\bar{\Omega}(x_0,r)} v(x), \quad m_1(\rho) := \operatorname*{ess\,inf}_{\bar{\Omega}(x_0,r)} v(x).
$$

Thus $\omega_1(\rho) = M_1(\rho) - m_1(\rho)$. Let $0 < r < 4r < \rho_0$. For an integer $n \geq 0$ we put

$$
K_n := M_1(4r) - 2^{-(n+1)}\omega_1(4r).
$$

Then K_n is an increasing sequence converging to $K_\infty = M_1(4r)$ and it is clear that $K_0 = \frac{1}{2}\left(M_1(4r) + m_1(4r)\right)$. Note that if v is a solution of (5.18) then $-v$ is also a solution of (5.18). By changing v to $-v$ if necessary, we may assume without restriction that $K_0 \geq 0$ and

$$
|A(K_0, 2r)| \leq \frac{1}{2}|\Omega(x_0, 2r)|. \tag{5.25}
$$

Thus $K_n \geq K_0 \geq 0$ for each integer n. By Proposition 5.2.5 (iii),

$$
|A(h,\rho)|^{\frac{2N-2}{N}} \leq c(h-k)^{-2}(R-\rho)^{-2} \int_{A(k,R)} |v-k|^2 \, dx \left(|A(k,\rho)| - |A(h,\rho)|\right)
$$

where $0 < \rho < R < \rho_0$ and all $h > k \geq 0$ for every $k \geq 0$ such that

$$
|A(k,\rho)| \leq \frac{1}{2}|\Omega(x_0,\rho)|.
$$

Since by (5.25) this condition is satisfied for $k = K_0$, taking $\rho = 2r$ and $R = 4r$, we obtain

$$
\begin{aligned}
|A(h,2r)|^{\frac{2N-2}{N}} &\leq c(h-k)^{-2}r^{-2} \int_{A(k,4r)} |v-k|^2 \, dx \left(|A(k,2r)| - |A(h,2r)|\right) \\
&\leq c(h-k)^{-2}r^{-2}|A(k,4r)|(M_1(4r)-k)^2 \\
&\quad \times \left(|A(k,2r)| - |A(h,2r)|\right) \\
&\leq c(h-k)^{-2}r^{N-2}(M_1(4r)-k)^2 \left(|A(k,2r)| - |A(h,2r)|\right)
\end{aligned}
$$

which implies that

$$\begin{aligned}\left(r^{-N}|A(h,2r)|\right)^{\frac{2N-2}{N}} &\leq c(h-k)^{-2}(M_1(4r)-k)^2\\ &\quad\times\left(r^{-N}|A(k,2r)|-r^{-N}|A(h,2r)|\right).\end{aligned}$$

Applying Lemma 5.1.4 with the function $\varphi(h) := r^{-N}|A(h,2r)|$ we obtain that

$$r^{-N}|A(h,2r)| \to 0 \qquad \text{as} \quad h \to M_1(4r).$$

Thus

$$r^{-N}|A(K_n,2r)| \to 0 \qquad \text{as} \quad K_n \to M_1(4r).$$

So we can choose n so large such that

$$|A(K_n,2r)| \leq \frac{1}{2}|\Omega(x_0,2r)|$$

and

$$Cr^{-N}|A(K_n,2r)| \leq \frac{1}{4}$$

where C is the constant in (5.24). The constant n can be chosen independtly of x_0. Now, using (5.24) with $k_0 = K_n$, we find that

$$\begin{aligned} M_1(r) &\leq K_n + \left(C(M_1(2r)-K_n)^2 r^{-N}|A(K_n,2r)|\right)^{1/2}\\ &\leq K_n + \frac{1}{2}(M_1(2r)-K_n)\\ &\leq K_n + \frac{1}{2}(M_1(4r)-K_n)\\ &= M_1(4r) - 2^{-(n+2)}\omega_1(4r).\end{aligned}$$

Since $m_1(r) \geq m_1(4r)$, the preceding inequality implies that

$$\begin{aligned}\omega_1(r) := M_1(r)-m_1(r) &\leq M_1(r)-m_1(4r)\\ &\leq M_1(4r)-2^{-(n+2)}\omega_1(4r)-m_1(4r)\\ &\leq (1-2^{-(n+2)})\omega_1(4r).\end{aligned}$$

Letting $\eta := (1-2^{-(n+2)})$, we obtain the inequality (5.20) and the claim is proved.

2) We claim that there exist two constants $K > 0$ and $0 < \alpha < 1$ such that if $w \in H^1_c(B(x_0,r))$ $(0 < r < \rho_0)$ is a solution of (5.19) we have

$$\omega_2(r) \leq Kr^{\alpha}, \tag{5.26}$$

where for $0 < r < \rho_0$,

$$\omega_2(r) := \operatorname*{ess\,sup}_{\bar\Omega(x_0,r)} w(x) - \operatorname*{ess\,inf}_{\bar\Omega(x_0,r)} w(x).$$

Indeed, for every real number $k \geq 0$, let $\varphi := (|w| - k)^+ \operatorname{sgn}(w)$. Then $\varphi \in H^1_c(B(x_0, r))$. Proceeding as in the proof of Proposition 5.2.1 and setting

$$H(k,r) := |A(k,r)|^{1/2-1/p+1/N} \|f\|_{p,\Omega} + [A(k,r) \cap \partial\Omega]^{1/s'-1/q} \|g\|_{q,\partial\Omega}$$

where $A(k,r) := \{x \in \bar{\Omega}(x_0, r) : \ |w(x)| \geq k\}$, we obtain that

$$\| \, |w| - k\|_{H^1(\Omega)} \leq cH(k,r).$$

It then follows that

$$\| \, |w| - k\|_{\frac{2N}{N-2}, A(k,r)} \leq c_1 H(k,r) \tag{5.27}$$

and

$$\| \, |w| - k\|_{s, A(k,r) \cap \partial\Omega} \leq c_2 H(k,r). \tag{5.28}$$

Let $h > k \geq 0$. The inequalities (5.27) and (5.28) imply that

$$|A(h,r)|^{\frac{N-2}{N}} \leq C_1 (h-k)^{-2} H(k,r)^2 \tag{5.29}$$

and

$$[A(k,r) \cap \partial\Omega]^{\frac{2}{s}} \leq C_2 (h-k)^{-2} H(k,r)^2. \tag{5.30}$$

Letting $a(h,r) := |A(h,r)| + [A(h,r) \cap \partial\Omega]^{\frac{2N}{s(N-2)}}$, we obtain that

$$a(h,r)^{\frac{N-2}{N}} \leq c(h-k)^{-2} H(k,r)^2.$$

Therefore we obtain the following estimate:

$$\begin{aligned} a(h,r)^{\frac{N-2}{N}} \quad &\leq \quad c(h-k)^{-2} \Big(|A(k,r)|^{1/N-1/p} \|f\|_p + \\ &\qquad + [A(k,r) \cap \partial\Omega]^{\frac{1}{s'} - \frac{1}{q} - \frac{N}{s(N-2)}} \|g\|_q \Big)^2 a(k,r). \end{aligned}$$

Since $|A(k,r)| \leq c \ r^N$ and since Ω has a Lipschitz boundary, there exist some constants $b_1, b_2 > 0$ such that

$$b_1 r^{N-1} \leq \sigma(\partial\Omega \cap B(x,r)) \leq b_2 r^{N-1} \quad \text{for} \ \ x \in \partial\Omega, \ 0 < r \leq 1,$$

(see [72, Theorem 4.14]), and it follows that

$$a(h,r)^{\frac{N-2}{N}} \leq c(h-k)^{-2} \left(r^{2(1-N/p)} \|f\|^2_{p,\Omega} + r^{-2(N-1)/q} \|g\|^2_{q,\partial\Omega} \right) a(k,r).$$

For r fixed, we let $\varphi(h) := a(h,r)^{\frac{N-2}{N}}$. Then

$$\varphi(h) \leq c(h-k)^{-2} \left(r^{2(1-N/p)} \|f\|^2_{p,\Omega} + r^{-2(N-1)/q} \|g\|^2_{q,\partial\Omega} \right) \varphi(k)^{\delta}$$

where $\delta = \frac{N}{N-2} = 1 + \frac{2}{N-2} > 1$. By Lemma 5.1.2, $\varphi(k_0 + d) = 0$. Taking $k_0 = 0$, we have $\varphi(d) = 0$ with

$$d^2 \leq c\left(r^{2(1-N/p)}\|f\|^2_{p,\Omega} + r^{-2(N-1)/q}\|g\|^2_{q,\partial\Omega}\right)\varphi(0)^{\delta-1}.$$

We therefore obtain,

$$\begin{aligned}\varphi(0)^{\delta-1} = a(0,r)^{\frac{2}{N}} &\leq \left(|A(0,r)| + [A(0,r)\cap\partial\Omega]^{\frac{2N}{s(N-2)}}\right)^{2/N}\\ &\leq c\left(r^N + r^{\frac{2N(N-1)}{s(N-2)}}\right)^{2/N}\\ &\leq c\, r^2.\end{aligned}$$

Finally we have

$$d^2 \leq C\left(r^{2(1-N/p)}\|f\|^2_{p,\Omega} + r^{2(1-(N-1)/q)}\|g\|^2_{q,\partial\Omega}\right).$$

This implies that

$$\operatorname*{ess\,sup}_{\bar{\Omega}(x_0,r)} w(x) \leq \operatorname*{ess\,sup}_{\bar{\Omega}(x_0,r)} |w(x)| \leq d \leq c\left(\|f\|^2_{p,\Omega} + \|g\|^2_{q,\partial\Omega}\right)^{1/2} r^\alpha \tag{5.31}$$

where $0 < \alpha = \min(1 - N/p, 1 - (N-1)/q) < 1$. As

$$\omega_2(r) := \operatorname*{ess\,sup}_{\bar{\Omega}(x_0,r)} w(x) - \operatorname*{ess\,inf}_{\bar{\Omega}(x_0,r)} w(x) \geq 0,$$

and since

$$\omega_2(r) \leq 2\operatorname*{ess\,sup}_{\bar{\Omega}(x_0,r)} |w(x)|,$$

by (5.31), we have the inequality (5.26) with $K := 2c\left(\|f\|^2_{p,\Omega} + \|g\|^2_{q,\partial\Omega}\right)^{1/2}$ and the claim is proved.

3) Next, let

$$\omega(r) := \operatorname*{ess\,sup}_{\bar{\Omega}(x_0,r)} u(x) - \operatorname*{ess\,inf}_{\bar{\Omega}(x_0,r)} u(x)$$

where $u = v + w$. Then

$$\omega(r) \leq \omega_1(r) + \omega_2(r).$$

By (5.20) and (5.26) we obtain that

$$\omega(r) \leq \eta\omega_1(4r) + Kr^\alpha \leq \eta\omega(4r) + Kr^\alpha.$$

Note that $0 < \eta < 1$ and K are independent of r. Since α depends only on $\Omega, \partial\Omega, N, p$ and q, by Lemma 5.1.5 and Remark 5.1.6, there exist two constants

$H > 0$ and $0 < \delta_1 < 1$ $(0 < \delta_1 \leq \alpha)$ where δ_1 depends only on $\Omega, \partial\Omega, N, p$ and q such that

$$\omega(r) \leq Hr^{\delta_1} \qquad \text{for} \quad 0 < r < \rho_0$$

which is the inequality (5.16).

4) Notice that it is well-known that a weak solution u of the problem (5.1) belongs to $C^1(\Omega)$. Finally the Hölder continuity on $\bar{\Omega}$ of the weak solution $u \in H^1(\Omega)$ of the problem (5.1) is a direct consequence of the inequality (5.16). $\square$

Now, let Δ_R be the selfadjoint operator on $L^2(\Omega)$ associated with the closed form $(a_\sigma, H^1(\Omega))$. Since Ω has a Lipschitz boundary, by Proposition 4.2.5, Δ_R is given by

$$\begin{cases} D(\Delta_R) = \{u \in H(\Delta, \Omega) : \ \left(\frac{\partial u}{\partial \nu} + \beta u\right)|_{\partial\Omega} = 0\} \\ \Delta_R u = \Delta u, \end{cases}$$

where we recall that

$$H(\Delta, \Omega) := \{u \in H^1(\Omega) : \ \Delta u \in L^2(\Omega)\}.$$

For a real number $\lambda \geq 0$, we denote by $R(\lambda, \Delta_R)$ the resolvent of Δ_R and for $u, v \in H^1(\Omega)$ we let

$$a_\sigma^\lambda(u, v) := a_\sigma(u, v) + \lambda \int_\Omega uv \, dx.$$

For $0 < \alpha < 1$, we set

$$C^{0,\alpha}(\bar{\Omega}) := \{u \in C(\bar{\Omega}) : \ \exists\, c > 0 : \ \forall\, x, y \in \Omega : \ |u(x) - u(y)| \leq c|x - y|^\alpha\}.$$

We obtain the following result as a corollary of Theorem 5.2.7.

Corollary 5.2.8. *The following assertions hold.*

a) Assume that $p > N$. Then for each $\lambda \geq 0$, $R(\lambda, \Delta_R)(L^p(\Omega)) \subset C^{0,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$ depending only on Ω, $\partial\Omega$, N and p.

b) For each $\lambda \geq 0$, $R(\lambda, \Delta_R)(C(\bar{\Omega}))$ is dense in $C(\bar{\Omega})$.

Proof. a) Let $\lambda \geq 0$, $f \in L^p(\Omega)$ and $u \in H^1(\Omega)$ satisfying

$$a_\sigma^\lambda(u, v) = \int_\Omega fv \, dx, \qquad \forall\, v \in H^1(\Omega). \tag{5.32}$$

If $\lambda = 0$, then u is a solution of (5.1) with $g = 0$ and by Theorem 5.2.7 $u \in C^{0,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$.

If $\lambda > 0$, the results of Chapter 4 or Proposition 5.2.1 (see also [34]) imply that the function u satisfying (5.32) is bounded. We obtain that u is a solution of the equation

$$a_\sigma(u, v) = \int_\Omega (f - \lambda u)v \, dx, \quad \forall v \in H^1(\Omega).$$

As $(f - \lambda u) \in L^p(\Omega)$, replacing f by $(f - \lambda u)$ in (5.1) and taking $g = 0$, Theorem 5.2.7 implies that $u \in C^{0,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$ which completes the proof of a).

b) Let $v \in C(\bar{\Omega})$. For every $\varepsilon > 0$, by Weierstrass Theorem, there exists $u \in C^\infty(\bar{\Omega})$ such that

$$\|v - u\|_\infty < \varepsilon.$$

For such $u \in C^\infty(\bar{\Omega})$, define the linear functional Lu on $H^1(\Omega)$ by: for every $\varphi \in H^1(\Omega)$ we let

$$\begin{aligned} \langle Lu, \varphi \rangle &:= -\int_\Omega \nabla u \nabla \varphi \, dx - \int_{\partial\Omega} \beta u \varphi \, d\sigma \\ &= -\sum_{i=1}^N \int_\Omega D_i u D_i \varphi \, dx - \int_{\partial\Omega} \beta u \varphi \, d\sigma. \end{aligned}$$

For each $\lambda \geq 0$, define $T := \lambda u - Lu$. Since T is a linear continuous functional on $H^1(\Omega)$, there exists a unique element $w \in H^1(\Omega)$ such that

$$a_\sigma^\lambda(w, \varphi) = \langle T, \varphi \rangle, \qquad \forall\, \varphi \in H^1(\Omega).$$

We denote this element w by $R_\lambda(\Delta_R)T$. In fact, one has $w := R_\lambda(\Delta_R)T = u$.

Next we show that there exists $g \in C^\infty(\bar{\Omega})$ such that

$$\|u - R(\lambda, \Delta_R)g\|_\infty < \varepsilon.$$

Let $k_i \in C^\infty(\bar{\Omega})$, $i = 0, \dots, N$ be such that

$$\|k_0 - \lambda u\|_p < \varepsilon \ \text{ and } \ \|D_i u - k_i\|_p < \frac{\varepsilon}{N}$$

for some $p > N$ and $i = 1, \dots, N$, and

$$\|\beta u - \sum_{i=1}^N k_i\|_{q,\partial\Omega} < \varepsilon$$

for some $q > N - 1$. Let

$$g := -k_0 + \sum_{i=1}^N D_i k_i.$$

Then $g \in C^\infty(\bar{\Omega})$. Moreover, for every $\varphi \in H^1(\Omega)$ we have

$$\begin{aligned} a_\sigma^\lambda(u - R(\lambda, \Delta_R)g, \varphi) &= a_\sigma^\lambda(u, \varphi) - a_\sigma^\lambda(R(\lambda, \Delta_R)g, \varphi) \\ &= \langle T, \varphi \rangle - \int_\Omega g\varphi \, dx \\ &= \int_\Omega (\lambda u - k_0)\varphi \, dx + \sum_{i=1}^N \int_\Omega (D_i u - k_i) D_i \varphi \, dx + \\ &\quad + \int_{\partial\Omega} \left(\beta u - \sum_{i=1}^N k_i \right) \varphi \, d\sigma. \end{aligned}$$

By Corollary 5.2.2,

$$\begin{aligned}\|u - R(\lambda, \Delta_R)g\|_\infty &\leq c\left(\|\lambda u - k_0\|_p + \sum_{i=1}^N \|D_i u - k_i\|_p + \|\beta u - \sum_{i=1}^N k_i\|_{q,\partial\Omega}\right)\\ &\leq 3c\varepsilon.\end{aligned}$$

Let $f \in C(\bar\Omega)$ be such that

$$\|g - f\|_\infty < \varepsilon.$$

Then

$$\begin{aligned}\|v - R(\lambda, \Delta_R)f\|_\infty &= \|v - u + u - R(\lambda, \Delta_R)g + R(\lambda, \Delta_R)g - R(\lambda, \Delta_R)f\|_\infty\\ &< \|v - u\|_\infty + \|u - R(\lambda, \Delta_R)g\|_\infty + \|R(\lambda, \Delta_R)(g - f)\|_\infty\\ &< C\varepsilon\end{aligned}$$

which completes the proof. □

Theorem 5.2.9. *Let $\Delta_R^{\bar\Omega}$ be the part of the operator Δ_R in $C(\bar\Omega)$ in the sense that*

$$D(\Delta_R^{\bar\Omega}) := \{u \in D(\Delta_R) \cap C(\bar\Omega) :\ \Delta_R u \in C(\bar\Omega)\}; \quad \Delta_R^{\bar\Omega} u := \Delta_R u.$$

Then $\Delta_R^{\bar\Omega}$ generates a holomorphic compact contractive C_0-semigroup $(T(t))_{t\geq 0}$ on $C(\bar\Omega)$.

Proof. By Corollary 5.2.8, the operator $\Delta_R^{\bar\Omega}$ generates a C_0-semigroup on $C(\bar\Omega)$ which is contractive.

The compactness follows from the formula

$$T(t) = ABCD$$

where $D = Id$ is bounded from $C(\bar\Omega)$ into $L^2(\Omega)$, $C = e^{t/3\Delta_R}$ is compact from $L^2(\Omega)$ into $L^2(\Omega)$, $B = e^{t/3\Delta_R}$ is bounded from $L^2(\Omega)$ into $L^\infty(\Omega)$ by ultracontractivity (see Chapter 4) and $A = e^{t/3\Delta_R}$ is bounded from $L^\infty(\Omega)$ into $C(\bar\Omega)$ by the **strong Feller property** (Corollary 5.2.8 a)).

Now we prove the holomorphy. Since the semigroup $(e^{t\Delta_R})_{t\geq 0}$ on $L^2(\Omega)$ is submarkovian, it induces contractive semigroups on all $L^p(\Omega)$, $1 \leq p \leq \infty$ which are strongly continuous if $1 \leq p < \infty$. Moreover, these semigroups are consistent (see Theorem 1.3.17). Let us denote by $\Delta_R^\infty := (\Delta_R^1)^*$ the generator of the semigroup on $L^\infty(\Omega)$. The consistence property and Corollary 5.2.8 a) imply that for each $\lambda \geq 0$,

$$R(\lambda, \Delta_R)(L^\infty(\Omega)) = R(\lambda, \Delta_R^\infty)(L^\infty(\Omega)) = D(\Delta_R^\infty) \subset C(\bar\Omega).$$

Thus $\Delta_R^{\bar\Omega}$ can also be defined as the part of the operator Δ_R^∞ in $C(\bar\Omega)$. We obtain that

$$D(\Delta_R^{\bar\Omega}) = \{u \in D(\Delta_R^\infty) \cap C(\bar\Omega) :\ \Delta_R^\infty u \in C(\bar\Omega)\} = D(\Delta_R^\infty).$$

Since $C(\bar{\Omega})$ is a closed subspace of $L^\infty(\Omega)$ and $D(\Delta_R^{\bar{\Omega}}) = D(\Delta_R^\infty)$ which is dense in $C(\bar{\Omega}) = \overline{D(\Delta_R^\infty)}$ (by Corollary 5.2.8 c)) and since $(e^{t\Delta_R^\infty})_{t\geq 0}$ is a holomorphic semigroup on $L^\infty(\Omega)$ (see [16, Theorem 5.2]), it follows from [11, Remark 3.7.13] that $(T(t))_{t\geq 0}$ is a holomorphic C_0-semigroup on $C(\bar{\Omega})$ which completes the proof. □

5.3 Resolvent Positive Operator.

In this section we define an operator A on $C(\bar{\Omega}) \times C(\partial\Omega)$ whose part in $C(\bar{\Omega})$ is the operator $\Delta_R^{\bar{\Omega}}$. Using an elliptic weak maximum principle, we will show that A is a resolvent positive operator.

Definition 5.3.1. *Let λ be a positive real number. A function $u \in H^1(\Omega)$ is called a subsolution of the equation associated with a_σ^λ if*

$$a_\sigma^\lambda(u, v) \leq 0 \quad \forall\, v \in H^1(\Omega), \quad \textit{such that}\, v \geq 0 \quad \textit{on} \quad \bar{\Omega} \quad \textit{in the “generalized sense”}. \tag{5.33}$$

We have the following elliptic weak maximum principle.

Proposition 5.3.2 (Elliptic weak maximum principle). *Let $u \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfy (5.33). Then $u \leq 0$ on $\bar{\Omega}$.*

Proof. Let $u \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfy the equation (5.33) and let $v := u^+$. Then $v \in H^1(\Omega) \cap C(\bar{\Omega})$ and $v \geq 0$ on $\bar{\Omega}$. As v is continuous on $\bar{\Omega}$, then $v \geq 0$ on $\bar{\Omega}$ in the “generalized sense” is the same than $v \geq 0$ on $\bar{\Omega}$ in the usual sense. Replacing v in (5.33), we obtain:

$$\lambda \int_{\{u>0\}} u^2 \, dx + \int_{\{u>0\}} |\nabla u|^2 \, dx + \int_{\{u>0\}\cap\partial\Omega} \beta u^2 \, d\sigma \leq 0.$$

Since $\beta(x) \geq \gamma > 0$ for some constant γ, it follows that for each $\lambda \geq 0$, we have that $u = 0$ or $u \leq 0$ a.e. on Ω and $u \leq 0$ σ a.e. on $\partial\Omega$. Since $u \in C(\bar{\Omega})$, this cleary implies that $u \leq 0$ on $\bar{\Omega}$. □

We obtain the following result as a consequence of the above elliptic weak maximum principle.

Proposition 5.3.3. *Let $f \in C(\bar{\Omega})$, $f \geq 0$, $\varphi \in C(\partial\Omega)$, $\varphi \geq 0$ and let $u \in H^1(\Omega)$ be a weak solution of the inhomogeneous Robin problem*

$$\begin{cases} \lambda u - \Delta u &= f \quad \textit{in} \;\; \Omega \\ \frac{\partial u}{\partial \nu} + \beta u &= \varphi \quad \textit{on} \;\; \partial\Omega \end{cases} \tag{5.34}$$

where $\lambda \geq 0$ is a real number. Then $u \geq 0$ on $\bar{\Omega}$.

Proof. We can decompose $u = v + w$, where $v \in H^1(\Omega)$ is a weak solution of the equation

$$\begin{cases} \lambda v - \Delta v & = f \quad \text{in } \Omega \\ \frac{\partial v}{\partial \nu} + \beta v & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{5.35}$$

and $w \in H^1(\Omega)$ is a weak solution of the equation

$$\begin{cases} \lambda w - \Delta w & = 0 \quad \text{in } \Omega \\ \frac{\partial w}{\partial \nu} + \beta w & = \varphi \quad \text{on } \partial\Omega. \end{cases} \tag{5.36}$$

1) It is well-known that $R(\lambda, \Delta_R)$ is a positive operator. Since $f \geq 0$ on $\bar{\Omega}$, it follows that $v := R(\lambda, \Delta_R) f \geq 0$ a.e. on Ω. As $v \in C(\bar{\Omega})$, it follows that $v \geq 0$ on $\bar{\Omega}$.

2) The solution w of (5.36) satisfies

$$a_\sigma^\lambda(w, \psi) = \int_{\partial\Omega} \varphi\psi \, d\sigma \qquad \forall\, \psi \in H^1(\Omega).$$

By Theorem 5.2.7, $w \in H^1(\Omega) \cap C(\bar{\Omega})$. As $\varphi \geq 0$ on $\partial\Omega$, then for all $\psi \in H^1(\Omega)$, $\psi \geq 0$ on $\bar{\Omega}$ in the "generalized sense" we have, $a_\sigma^\lambda(w, \psi) \geq 0$. One obtains that $a_\sigma^\lambda(-w, \psi) = -a_\sigma^\lambda(w, \psi) \leq 0$ and then $(-w)$ is a subsolution of the equation associated with a_σ^λ. By Proposition 5.3.2, $-w \leq 0$ on $\bar{\Omega}$ and thus $w \geq 0$ on $\bar{\Omega}$. Finally, one obtains that $u = v + w \geq 0$ on $\bar{\Omega}$. $\square$

On $C(\bar{\Omega})$ we consider the operator Δ_m defined by

$$\begin{cases} D(\Delta_m) & := \{u \in H^1(\Omega) \cap C(\bar{\Omega}) : \ \Delta u \in C(\bar{\Omega}) : \ \exists\, \varphi \in C(\partial\Omega) : \\ & \quad a_\sigma(u, v) = -\int_\Omega \Delta u v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma, \ \forall\, v \in H^1(\Omega)\} \\ \Delta_m u & := \Delta u \quad \text{in } \ \Omega. \end{cases}$$

Obviously, Δ_m is a closed operator on $C(\bar{\Omega})$.

Next we consider the operator A on $C(\bar{\Omega}) \times C(\partial\Omega)$ defined by

$$\begin{cases} D(A) & := D(\Delta_m) \times \{0\}, \\ A(u, 0) & := (\Delta u, -\varphi), \end{cases}$$

where φ si given by the condition $u \in D(\Delta_m)$ and the Banach space $C(\bar{\Omega}) \times C(\partial\Omega)$ is equipped with the norm

$$\|(f, g)\|_{C(\bar{\Omega}) \times C(\partial\Omega)} := \max\left(\|f\|_{C(\bar{\Omega})}, \|g\|_{C(\partial\Omega)}\right).$$

Proposition 5.3.4. *Let $f \in C(\bar{\Omega})$ and $g \in C(\partial\Omega)$. Then $-A(u, 0) = (f, g)$ if and only if u is a weak solution of the problem (5.1).*

Proof. Let $u \in D(\Delta_m)$ be such that $-A(u,0) = (f,g)$. Then

$$\begin{cases} -\Delta u &= f \qquad \text{in} \quad \Omega \\ \varphi &= g \qquad \text{on} \quad \partial\Omega; \end{cases}$$

where φ is the function given by the condition $u \in D(\Delta_m)$. We obtain that for every $v \in H^1(\Omega)$,

$$\begin{aligned} \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \beta uv \, d\sigma &= -\int_\Omega \Delta uv \, dx + \int_{\partial\Omega} \varphi v \, d\sigma \\ &= \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, d\sigma \end{aligned}$$

and thus u is a weak solution of (5.1).

To prove the converse, let $u \in H^1(\Omega)$ be a weak solution of (5.1). By Theorem 5.2.7, $u \in C(\bar{\Omega})$. Since $\Delta u = -f \in C(\bar{\Omega})$, taking $\varphi = g \in C(\partial\Omega)$, one has that $u \in D(\Delta_m)$ and $-A(u,0) = (f,g)$. □

Next, let X be a Banach lattice. A closed operator B on X is called a **resolvent positive operator**, if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(B)$ (**the resolvent set** of B) and $R(\lambda, B) \geq 0$ for all $\lambda > \omega$. We denote by

$$s(B) = \sup\{\operatorname{Re}\lambda : \ \lambda \in \sigma(B)\}$$

the **spectral bound** of B, where $\sigma(B)$ denotes the spectrum of B.

Proposition 5.3.5. *The operator A is resolvent positive and $s(A) < 0$.*

Proof. a) Let $\lambda \geq 0$ be a real number and suppose that $\lambda \in \rho(A)$. Let $f \in C(\bar{\Omega})$, $g \in C(\partial\Omega)$ and $(u,0) = R(\lambda, A)(f,g)$. Then u is a solution of the equation

$$a_\sigma^\lambda(u,v) = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, d\sigma, \ \ \forall \, v \in H^1(\Omega).$$

If $f \leq 0$ and $g \leq 0$, it follows from the weak maximum principle that $u \leq 0$ in $\bar{\Omega}$. Thus $R(\lambda, A) \geq 0$.

b) We show that $0 \in \rho(A)$. Let $f \in C(\bar{\Omega})$ and $g \in C(\partial\Omega)$. Let $u \in H^1(\Omega)$ be a weak solution of (5.1). Taking $\varphi = g$, we have that $u \in D(\Delta_m)$ and $-A(u,0) = (f,g)$. Thus $-A$ is surjective. Since a weak solution of (5.1) is unique, it follows that $-A$ is bijective and $0 \in \rho(A)$.

c) Proceeding exactly as in the proof of [11, Theorem 6.1.6 c)], we obtain that $[0, \infty) \subset \rho(A)$ and this completes the proof. □

Proposition 5.3.6. *The operator $\Delta_R^{\bar{\Omega}}$ is the part of A in $C(\bar{\Omega})$. Moreover we have $\rho(A) = \rho(\Delta_R^{\bar{\Omega}})$.*

Proof. a) We claim that $\overline{D(A)} = C(\bar{\Omega}) \times \{0\}$.

(i) First, let

$$D := \{u \in H^1(\Omega) \cap C(\bar{\Omega}) : \ \Delta u \in C(\bar{\Omega}), \ \left(\frac{\partial u}{\partial \nu} + \beta u\right) \in C(\partial\Omega)\}.$$

We show that $D(\Delta_m) = D$. Indeed, let $u \in D$ and $\varphi := \frac{\partial u}{\partial \nu} + \beta u$. Then integrating by parts yields,

$$\begin{aligned} -\int_\Omega \Delta u v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma &= \int_\Omega \nabla u \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma + \int_{\partial\Omega} \varphi v \, d\sigma \\ &= a_\sigma(u, v) \qquad \forall \, v \in H^1(\Omega). \end{aligned}$$

Thus $u \in D(\Delta_m)$.

To prove the converse inclusion, let $u \in D(\Delta_m)$. By hypothesis, $u \in H^1(\Omega) \cap C(\bar{\Omega})$, $\Delta u \in C(\bar{\Omega})$ and there exists $\varphi \in C(\partial\Omega)$ such that

$$a_\sigma(u, v) = -\int_\Omega \Delta u v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma, \quad \forall \, v \in H^1(\Omega).$$

Integrating by parts yields,

$$\int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \beta u v \, d\sigma = \int_\Omega \nabla u \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma + \int_{\partial\Omega} \varphi v \, d\sigma.$$

This implies that

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} + \beta u\right) v \, d\sigma = \int_{\partial\Omega} \varphi v \, d\sigma \quad \forall \, v \in H^1(\Omega).$$

Since $\varphi \in C(\partial\Omega)$, this gives that $(\frac{\partial u}{\partial \nu} + \beta u) \in C(\partial\Omega)$ and thus $u \in D$.

(ii) We claim that $D(\Delta_R^{\bar{\Omega}}) \subset D(\Delta_m)$. In fact, let $u \in D(\Delta_R^{\bar{\Omega}})$. By definition, $u \in C(\bar{\Omega}) \cap D(\Delta_R)$ and $\Delta_R u \in C(\bar{\Omega})$ where we recall that

$$D(\Delta_R) := \{u \in H(\Delta, \Omega) : \ \left(\frac{\partial u}{\partial \nu} + \beta u\right) = 0\}.$$

We obtain that $u \in H^1(\Omega) \cap C(\bar{\Omega})$, $\Delta u \in C(\bar{\Omega})$ and $\left(\frac{\partial u}{\partial \nu} + \beta u\right) = 0 \in C(\partial\Omega)$. Thus $u \in D(\Delta_m)$.

(iii) Since $D(\Delta_R^{\bar{\Omega}}) \subset D(\Delta_m)$ and $D(\Delta_R^{\bar{\Omega}})$ is dense in $C(\bar{\Omega})$ this implies that $D(\Delta_m)$ is dense in $C(\bar{\Omega})$ and the proof of a) is complete.

b) Let $A_{\bar{\Omega}}$ be the part of A in $C(\bar{\Omega}) \times \{0\}$. Then

$$\begin{cases} D(A_{\bar{\Omega}}) := \{(u, 0) \in D(A) \cap \big(C(\bar{\Omega}) \times \{0\}\big) : \ A(u, 0) \in C(\bar{\Omega}) \times \{0\}\}, \\ A_{\bar{\Omega}}(u, 0) = A(u, 0). \end{cases}$$

Let $(u,0) \in D(A_{\bar{\Omega}})$. The condition $A(u,0) \in C(\bar{\Omega}) \times \{0\}$ means that $-A(u,0) = (f,0)$ for some $f \in C(\bar{\Omega})$. By Proposition 5.3.4, this implies that u is a weak solution of the equation (5.1). We obtain that

$$-A(u,0) = (-\Delta u, (\frac{\partial u}{\partial \nu} + \beta u)) = (f,0).$$

Thus

$$D(A_{\bar{\Omega}}) = \{(u,0) \in D(A) \cap \big(C(\bar{\Omega}) \times \{0\}\big) : \ (\Delta u, (\frac{\partial u}{\partial \nu} + \beta u)) \in C(\bar{\Omega}) \times \{0\}\}.$$

Identifying $C(\bar{\Omega}) \times \{0\}$ with $C(\bar{\Omega})$, one obtains that

$$D(A_{\bar{\Omega}}) = \{u \in C(\bar{\Omega}) \cap D(\Delta_R) : \ \Delta u \in C(\bar{\Omega})\} = D(\Delta_R^{\bar{\Omega}}).$$

Moreover, with this identification, $A_{\bar{\Omega}} u = \Delta u$ and the proof of this part is complete.

c) Now we show that $\rho(A) = \rho(\Delta_R^{\bar{\Omega}})$. Let $\lambda \in \rho(A)$, $f \in C(\bar{\Omega})$ and $(u,0) = R(\lambda, A)(f,0)$. Then $(\lambda - A)(u,0) = (f,0)$ and u is a solution of the equation (5.32) which is in $H^1(\Omega) \cap C(\bar{\Omega})$. This gives that $R(\lambda, A)(C(\bar{\Omega}) \times \{0\}) \subset C(\bar{\Omega}) \times \{0\}$ and thus $\rho(A) \subset \rho(\Delta_R^{\bar{\Omega}})$.

To prove the converse inclusion, let $\lambda \in \rho(\Delta_R^{\bar{\Omega}})$. For $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$, let $u \in H^1(\Omega) \cap C(\bar{\Omega})$ be a solution of the equation (5.34). We obtain that $(\frac{\partial u}{\partial \nu} + \beta u) := \varphi \in C(\partial\Omega)$ and

$$a_\sigma(u,v) = -\int_\Omega \Delta u v \, dx + \int_{\partial\Omega} \varphi v \, d\sigma$$

for every $v \in H^1(\Omega)$. This implies that $u \in D(\Delta_m)$ and $(\lambda - \Delta)u = f$. Then $(u,0) \in D(A)$, $(\lambda - A)(u,0) = (f,\varphi)$ and thus $(\lambda - A)$ is surjective. Let $u \in D(\Delta_m)$ be such that $(\lambda - A)(u,0) = (0,0)$. Then u is a solution of the equation (5.34) with $f = 0$ and $\varphi = 0$. We obtain that $u \in D(\Delta_R^{\bar{\Omega}})$ and $(\lambda - \Delta_R^{\bar{\Omega}})u = 0$. Thus $u = 0$ and the proof is complete. □

Remark 5.3.7. *If follows from the proof of the preceding proposition that for $u \in D(\Delta_m)$, we have*

$$-A(u,0) = (-\Delta u, (\frac{\partial u}{\partial \nu} + \beta u)).$$

Finally, notice that the fact that the operator A is a resolvent positive operator can be used to study the well-posedness of the heat equation with inhomogeneous boundary conditions

$$\begin{cases} u'(t) = \Delta u(t) & (t \in [0,\tau]); \\ \Big(\frac{\partial u(t)}{\partial \nu} + \beta u(t)\Big)\,|_{\partial\Omega} = \varphi(t), & (t \in [0,\tau]; \\ u(0) = u_0, \end{cases}$$

where $u_0 \in C(\bar{\Omega})$ and $\varphi \in C([0,\tau], C(\partial\Omega))$ are given. The case of the inhomogeneous Dirichlet boundary conditions is contained in [11, Chapter 6]. We will not go into details.

Chapter 6

Wentzell-Robin Boundary Conditions on $C[0, 1]$

In this chapter we consider the operator A_W on $C[0, 1]$ defined by

$$\begin{cases} D(A_W) := \{u \in C^2[0,1] : \, (au')'(j) + \beta_j u'(j) + \gamma_j u(j) = 0; \; j = 0, 1\} \\ A_W u := (au')', \end{cases} \tag{6.1}$$

where $\beta_j, \; \gamma_j \; (j = 0, 1)$ are arbitrary real numbers and the function $a \in C^1[0, 1]$ satisfies

$$a(x) \geq \alpha > 0 \tag{6.2}$$

for some real number α. We call the operator A_W **the realization of the operator** $(au')'$ **on** $C[0, 1]$ **with Wentzell-Robin boundary conditions**. Note that this boundary condition called Wentzell-Robin boundary condition is a dynamic boundary condition (see [14]). We will use perturbation arguments to show that A_W generates a holomorphic C_0-semigroup.

6.1 Intermediate Results.

Before starting the study of the problem mentioned in the introduction, we prove the following abstract result which we shall use frequently.

Theorem 6.1.1. *Let Y be a Banach space and A_Y be a generator of a holomorphic C_0-semigroup $T_Y = (T_Y(t))_{t \geq 0}$ on Y. Let $X := Y \oplus Z$ for some Banach space Z and A be the closed operator defined on X by*

$$Ax = A(y, z) := (A_Y y, 0)$$

with domain $D(A) = D(A_Y) \oplus Z$. Then A generates a holomorphic C_0-semigroup on X.

Proof. Note that A_Y is the part of A in Y. It is also easy to see that A generates a C_0-semigroup $T = (T(t))_{t\geq 0}$ on X and for every $x := (y,z) \in X$, we have $T(t)x = (T_Y(t)y, z)$. Since $T_Y = (T_Y(t))_{t\geq 0}$ is a holomorphic semigroup on Y, it follows that $T = (T(t))_{t\geq 0}$ is a holomorphic semigroup on X. □

Next, we consider the bilinear form $l:\ H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$ on $L^2(0,1)$ defined by

$$l(u,v) := \int_0^1 a(x)u'(x)v'(x)\,dx + \int_0^1 b(x)u'(x)v(x)\,dx + \int_0^1 c(x)u(x)v(x)\,dx;$$

where we assume for the moment that $a \in L^\infty(0,1)$ satisfies the condition (6.2) and $b, c \in C[0,1]$.

It is well-known (see [12]) that $(l, H_0^1(0,1))$ is a bilinear densely defined closed form on $L^2(0,1)$ which is also **elliptic**; i.e., there exist $\omega \in \mathbb{R}$ and $\mu > 0$ such that

$$l(u,u) + \omega\|u\|^2_{L^2(0,1)} \geq \mu\|u\|^2_{H^1(0,1)}$$

for every $u \in H_0^1(0,1)$. Let L_2 be the closed operator on $L^2(0,1)$ associated with the form $(l, H_0^1(0,1))$; i.e.,

$$\begin{cases} D(L_2) := \{u \in H_0^1(0,1):\ \exists\, v \in L^2(0,1):\ l(u,\varphi) = (v,\varphi)\ \forall\, \varphi \in H_0^1(0,1)\} \\ L_2 u := -v, \end{cases}$$

where $(\ ,\ )$ denotes the scalar product in $L^2(0,1)$. It is easy to see that the operator L_2 is given by

$$\begin{cases} D(L_2) = \{u \in H_0^1(0,1):\ (au')' - bu' - cu \in L^2(0,1)\} \\ L_2 u = (au')' - bu' - cu. \end{cases}$$

Moreover, since $H_0^1(0,1)$ is dense in $L^2(0,1)$ and $(l, H_0^1(0,1))$ is a bilinear closed elliptic form, it follows from Theorem 1.3.4 that L_2 generates a holomorphic C_0-semigroup $T_2 = (T_2(t))_{t\geq 0}$ on $L^2(0,1)$. It is also easy to verify that the first Beurling-Deny criterion for non-symmetric forms is satisfied (see [80, Théorème 1.2.2]). Thus the semigroup is positive. If $c \geq 0$, then the second Beurling-Deny criterion is satisfied (see [12, Section 4] or [80, Théorème 1.2.5]) and this implies that the semigroup is submarkovian.

Next let L_0 be the part of L_2 in $C_0(0,1) := \{u \in C[0,1]:\ u(0) = u(1) = 0\}$; i.e.,

$$\begin{cases} D(L_0) := \{u \in D(L_2) \cap C_0(0,1):\ (au')' - bu' - cu \in C_0(0,1)\} \\ L_0 u := L_2 u = (au')' - bu' - cu. \end{cases}$$

Proposition 6.1.2. *a) Assume that $b \in C^1[0,1]$. Then L_0 generates a holomorphic C_0-semigroup on $C_0(0,1)$.*

b) Assume that $a \in C^1[0,1]$. Then L_0 is given by

$$\begin{cases} D(L_0) = \{u \in C^2[0,1] \cap C_0(0,1) : \ (au')' - bu' - cu \in C_0(0,1)\} \\ L_0 u = (au')' - bu' - cu. \end{cases}$$

Proof. a) This part follows from [12, Corollary 4.7].
b) Set

$$D := \{u \in C^2[0,1] \cap C_0(0,1) : \ (au')' - bu' - cu \in C_0(0,1)\}.$$

Let $u \in D$. Then $u \in H_0^1(0,1)$ and $(au')' - bu' - cu \in L^2(0,1)$. We obtain that $u \in D(L_2) \cap C_0(0,1)$ and $(au')' - bu' - cu \in C_0(0,1)$. Thus $u \in D(L_0)$.

To prove the converse inclusion, let $u \in D(L_0)$. Then u is a solution of the Dirichlet problem

$$\begin{cases} u \in H_0^1(0,1) \\ (au')' - bu' - cu = f, \end{cases}$$

where $f \in C_0(0,1) \subset L^\infty(0,1)$. Since we assume that $a \in C^1[0,1]$, it follows from [37, Proposition 13 p. 605] that $u \in C^1[0,1]$ and this implies that $(au')' \in C[0,1]$. As $(au')' = a'u' + au''$ then $au'' \in C[0,1]$. The continuity of a and the fact that $a(x) \geq \alpha > 0$ imply that $u'' \in C[0,1]$ and then $u \in C^2[0,1]$. Finally, we obtain that $u \in C^2[0,1] \cap C_0(0,1)$ and $(au')' - bu' - cu \in C_0(0,1)$ which completes the proof. $\square$

Throughout the following, we shall assume that $c \in C[0,1]$ satisfies $c(x) \geq 0$ and $a, b \in C^1[0,1]$ and that the function a also satisfies the condition (6.2).

Next we define an operator A on $C[0,1]$ by

$$\begin{cases} D(A) := \{u \in C^2[0,1] : \ (au')' - bu' - cu \in C_0(0,1)\} \\ Au := (au')' - bu' - cu. \end{cases} \tag{6.3}$$

Proposition 6.1.3. *The operator A generates a holomorphic C_0-semigroup on $C[0,1]$.*

Proof. a) We claim that A is a closed operator on $C[0,1]$. Indeed, it is well-known (see [25, Remark 9 p.133]) that for $u \in C^2[0,1]$,

$$\|u\| := \|u\|_\infty + \|u''\|_\infty$$

is a norm on $C^2[0,1]$ which is equivalent to the norm of $C^2[0,1]$ given by:

$$\|u\|_{C^2[0,1]} := \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty.$$

Moreover, using Taylor's formula (see [25, Remark 9 p.133]), we obtain that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for every $u \in C^2[0,1]$,

$$\|u'\|_\infty \leq \varepsilon \|u''\|_\infty + C_\varepsilon \|u\|_\infty. \tag{6.4}$$

We finally obtain that for every $u \in D(A) \subset C^2[0,1]$,

$$\begin{aligned}
\|u'\|_\infty &\leq \varepsilon\|u''\|_\infty + C_\varepsilon\|u\|_\infty \\
&\leq \varepsilon\alpha^{-1}\|au''\|_\infty + C_\varepsilon\|u\|_\infty \\
&\leq \varepsilon\alpha^{-1}\|au'' + a'u' - a'u' - bu' - cu + bu' + cu\|_\infty + C_\varepsilon\|u\|_\infty \\
&\leq \varepsilon\alpha^{-1}\|Au\|_\infty + \varepsilon\alpha^{-1}(\|a'\|_\infty + \|b\|_\infty)\|u'\|_\infty + \\
&\quad +(\varepsilon\alpha^{-1}\|c\|_\infty + C_\varepsilon)\|u\|_\infty.
\end{aligned}$$

From this inequality, we see that there exist some constants $k_1, k_2 \geq 0$ such that for every $u \in D(A)$,

$$\|u'\|_\infty \leq k_1\|Au\|_\infty + k_2\|u\|_\infty. \tag{6.5}$$

Using the inequality (6.4), we obtain the following estimates.

$$\begin{aligned}
\|u''\|_\infty &\leq \alpha^{-1}\|(au')' - bu' - cu - a'u' + bu' + cu\|_\infty \\
&\leq \alpha^{-1}\|Au\|_\infty + \alpha^{-1}(\|a'\|_\infty + \|b\|_\infty)\|u'\|_\infty + \alpha^{-1}\|c\|_\infty\|u\|_\infty \\
&\leq \alpha^{-1}\|Au\|_\infty + \varepsilon\alpha^{-1}(\|a'\|_\infty + \|b\|_\infty)\|u''\|_\infty + \\
&\quad +\alpha^{-1}(C_\varepsilon(\|a'\|_\infty + \|b\|_\infty) + \|c\|_\infty)\|u\|_\infty.
\end{aligned}$$

Choosing ε suitably, we obtain that there exist some constants $c_1, c_2 > 0$ such that for every $u \in D(A)$,

$$\|u''\|_\infty \leq c_1\|Au\|_\infty + c_2\|u\|_\infty. \tag{6.6}$$

Let $u_n \in D(A)$ and $u, v \in C[0,1]$ be such that

$$\lim_{n\to\infty} \|u_n - u\|_\infty = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Au_n - v\|_\infty = 0.$$

It follows from (6.5) and (6.6) that u_n' and u_n'' are Cauchy sequences in $C[0,1]$ and then converge uniformly. Thus $u \in C^2[0,1]$. Since $Au_n(0) = Au_n(1) = 0$ and Au_n converges to v uniformly, it follows that $v(0) = v(1) = 0$ and then $u \in D(A)$ and $Au = v$.

b) We claim that $C[0,1] = C_0(0,1) \oplus \ker(A)$. In fact, let $f \in C[0,1]$ and $u \in H^1(0,1)$ be a solution of the Dirichlet problem

$$\begin{cases} Lu := (au')' - bu' - cu = 0 \\ u(0) = f(0) \\ u(1) = f(1). \end{cases} \tag{6.7}$$

By [59, Theorem 8.34], one has $u \in C^2[0,1]$ and then $u \in \ker(A)$. Writing $f = (f-u) + u$, we obtain that $(f-u) \in C_0(0,1)$ and $u \in \ker(A)$. Since by [59, Corollary 8.2], the equation

$$\begin{cases} u \in H_0^1(0,1) \\ Lu = 0 \end{cases}$$

has only the solution $u = 0$, the decomposition is unique.

c) We claim that $D(A) = D(L_0) \oplus \ker(A)$. Indeed, let $f \in D(A)$ and $u \in C^2[0,1]$ be a solution of the equation (6.7). Then $(f-u) \in C^2[0,1] \cap C_0(0,1)$ and $L(f-u) = Lf - Lu = Lf \in C_0(0,1)$. Thus $(f-u) \in D(L_0)$. Moreover, $f = (f-u)+u$ and $u \in \ker(A)$. The uniqueness of the decomposition is obtained exactly as in b).

d) Note that the decompositions in b) and in c) are also topological. Moreover, it is clear that L_0 is the part of A in $C_0(0,1)$. Since, by Proposition 6.1.2, L_0 generates a holomorphic C_0-semigroup on $C_0(0,1)$, it follows from Theorem 6.1.1 that A generates a holomorphic C_0-semigroup on $C[0,1]$ and the proof is complete. □

The fact that A generates a C_0-semigroup on $C[0,1]$ (without holomorphy property) can be also obtained from the results of Clément and Timmermans in [30] with a different proof.

6.2 Holomorphy.

Throughout this section, $(A, D(A))$ will denote the operator defined by (6.3) in Section 6.1.

Let A_W be the operator defined in (6.1). The following is the main result of this chapter.

Theorem 6.2.1. *The operator A_W generates a holomorphic C_0-semigroup on $C[0,1]$.*

Proof. Let B be the operator on $C[0,1]$ defined by

$$D(B) := C^1[0,1] \quad \text{and} \quad Bu := bu' + cu.$$

Then $D(A) \subset D(B)$. For every $u \in D(A)$ we have

$$\|Bu\|_\infty := \|bu' + cu\|_\infty \le \|b\|_\infty \|u'\|_\infty + \|c\|_\infty \|u\|_\infty.$$

Using the inequality (6.4), we obtain that for every $\varepsilon > 0$,

$$\begin{aligned}
\|Bu\|_\infty &\le \|b\|_\infty(\varepsilon\|u''\|_\infty + C_\varepsilon\|u\|_\infty) + \|c\|_\infty\|u\|_\infty \\
&\le \varepsilon\|b\|_\infty\|u''\|_\infty + (C_\varepsilon\|b\|_\infty + \|c\|_\infty)\|u\|_\infty \\
&\le \alpha^{-1}\varepsilon\|b\|_\infty\|(au')' - bu' - cu - a'u' + bu' + cu\|_\infty + \\
&\quad +(C_\varepsilon\|b\|_\infty + \|c\|_\infty)\|u\|_\infty \\
&\le \alpha^{-1}\varepsilon\|b\|_\infty\|Au\|_\infty + \alpha^{-1}\varepsilon\|b\|_\infty\|a'\|_\infty\|u'\|_\infty + \\
&\quad +\alpha^{-1}\varepsilon\|b\|_\infty\|Bu\|_\infty + (C_\varepsilon\|b\|_\infty + \|c\|_\infty)\|u\|_\infty.
\end{aligned}$$

Using the inequality (6.5), we obtain the following estimates for every $\varepsilon > 0$ and every $u \in D(A)$.

$$\begin{aligned}\|Bu\|_\infty \quad \leq \quad & \alpha^{-1}\varepsilon\|b\|_\infty\|Au\|_\infty + \alpha^{-1}\varepsilon\|a'\|_\infty\|b\|_\infty\left(k_1\|Au\|_\infty + k_2\|u\|_\infty\right) + \\ & +\alpha^{-1}\varepsilon\|b\|_\infty\|Bu\|_\infty + (C_\varepsilon\|b\|_\infty + \|c\|_\infty)\|u\|_\infty.\end{aligned}$$

Thus

$$\|Bu\|_\infty \leq \varepsilon C_1\|Au\|_\infty + \varepsilon C_2\|Bu\|_\infty + C_3\|u\|_\infty, \tag{6.8}$$

where

$$C_1 := \alpha^{-1}\|b\|_\infty\left(1 + k_1\|a'\|_\infty\right), \quad C_2 := \alpha^{-1}\|b\|_\infty$$

and

$$C_3 := \varepsilon k_2\alpha^{-1}\|a'\|_\infty\|b\|_\infty + C_\varepsilon\|b\|_\infty + \|c\|_\infty.$$

Let $\omega \in \mathbb{R}$. Since by Proposition 6.1.3, A generates a holomorphic C_0-semigroup on $C[0,1]$, then the operator $(A-\omega)$ generates a bounded holomorphic C_0-semigroup on $C[0,1]$ for ω sufficiently large. This implies that there exist $\theta \in (0,\pi/2]$ and $M \geq 1$ such that for every

$$\lambda \in \Sigma_{\frac{\pi}{2}+\theta} := \{z \in \mathbb{C}\setminus\{0\} : \ |\arg z| \leq \frac{\pi}{2} + \theta\},$$

we have

$$\lambda \in \rho(A-\omega) \ \text{ and } \ \|R(\lambda, A-\omega)\| \leq \frac{M}{|\lambda|}.$$

From the inequality (6.8), we obtain that

$$\|Bu\|_\infty \leq \varepsilon C_1\|(A-\omega)u\|_\infty + \varepsilon C_2\|Bu\|_\infty + (\varepsilon C_1|\omega| + C_3)\|u\|_\infty.$$

Choosing ε such that

$$0 < \frac{\varepsilon C_1}{1-\varepsilon C_2} < \frac{1}{M+1},$$

the Perturbation Theorem in [42, Chap. III Theorem 2.13] implies that the operator $((A+B-\omega), D(A))$ generates a bounded holomorphic C_0-semigroup on $C[0,1]$. Therefore the operator A_0 defined by

$$D(A_0) = D(A) \quad \text{and} \quad A_0u = Au + Bu$$

generates a holomorphic C_0-semigroup on $C[0,1]$. The operator A_0 is given by

$$\begin{cases} D(A_0) & = \{u \in C^2[0,1] : \ (au')'(j) - b(j)u'(j) - c(j)u(j) = 0; \ j = 0,1\} \\ A_0u & = (au')'. \end{cases}$$

Setting $\beta_j := -b(j)$ and $\gamma_j := -c(j)$ for $j = 0,1$, we obtain that $D(A_0) = D(A_W)$ and $A_Wu = A_0u = (au')'$. Finally, note that, since $c(x) \geq 0$, then $\gamma_j := -c(j) \leq 0$.

The case $\gamma_j \geq 0$ follows from [60, Theorem 2.4] by perturbing the boundary conditions of the operator A_0 by the compact operator

$$\Phi : \; C[0,1] \to \mathbb{R}^2 : \; u \mapsto (-2c(0)u(0), -2c(1)u(1))$$

and by setting $\beta_j := -b(j)$ and $\gamma_j := c(j)$ for $j = 0, 1$ which completes the proof. $\square$

Bibliography

[1] T. Abe and K. Gustafson. The third boundary conditions - Was it Robin's? *The Mathematical Intelligencer* 20 (1998), 63–71.

[2] D. R. Adams and L. I. Hedberg. *Function Spaces and Potential Theory.* Springer-Verlag, Berlin, 1996.

[3] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[4] L. Ambrosion, N. Fusco and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Science Publications, Oxford, 2000.

[5] C. J. Amick. Some remarks on Rellich's theorem and the Poincaré inequality. *J. London Math. Soc.* 18 (1978), 81–93.

[6] W. Arendt. Semigroup properties by Gaussian estimates. *RIMS, Kyoto, Kokyuroku* 1009 (1997), 162–180.

[7] W. Arendt. Does diffusion determine the body? *J. Reine und Angewandte Math.*, to appear.

[8] W. Arendt and C. J. K. Batty. L'holomorphie du semi-groupe engendré par le Laplacien Dirichlet sur $L^1(\Omega)$. *C. R. Acad. Sci. Paris, Sér. I Math.* 315 (1992), 31–35.

[9] W. Arendt and C. J. K. Batty. Exponential stability of a diffusion equation with absorption. *Diff. Int. Equ.* 6 (1993), 1009–1024.

[10] W. Arendt and C. J. K. Batty. Absorption semigroups and Dirichlet boundary conditions. *Math. Ann.* 295 (1993), 427–448.

[11] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems.* Birkhäuser, Basel, 2001.

[12] W. Arendt and Ph. Bénilan. Wiener regularity and heat semigroups on spaces of continuous functions. *Topics in Nonlinear Analysis, J. Escher, G. Simonett eds., Birkhäuser, Basel* (1998), 29–49.

[13] W. Arendt and A. V. Bukhvalov. Integral representations of resolvents and semigroups. *Forum Math.* 6 (1994), 111–135.

[14] W. Arendt, G. Metafune, D. Pallara and S. Romanelli. The Laplacian with Wentzell-Robin boundary conditions on spaces of continuous functions. *Prepint,* 2002.

[15] W. Arendt and S. Monniaux. Domain perturbation for the first eigenvalue of the Dirichlet Schrödinger operator. *Operator Theory: Advances and Appl.* 78 (1995), 9–19.

[16] W. Arendt and A. F. M. Ter Elst. Gaussian estimates for second order elliptic operators with boundary conditions. *J. Operator Theory* 38 (1997), 87–130.

[17] W. Arendt and M. Warma. The Laplacian with Robin boundary conditions on arbitrary domains. *Preprint*, (2002).

[18] W. Arendt and M. Warma. Dirichlet and Neumann boundary conditions: What is in between? *Preprint*, (2002).

[19] D. H. Armitage and S. J. Gardiner. *Classical Potential Theory.* Springer-Verlag, Berlin, 2001.

[20] C. J. K. Batty. Asymptotic stability of Schrödinger semigroups: path integral methods. *Math. Ann.*, 292 (1992), 457–492.

[21] J. Baxter, G. Dal Maso and U. Mosco. Stopping times and Γ-convergence. *Trans. Amer. Math. Soc.* 303 (1987), 1–38.

[22] M. Biegert. The Neumann Laplacian on spaces of continuous functions. *Ulmer Seminare* 5 (2000), 137–147.

[23] N. Bouleau and F. Hirsch. *Dirichlet Forms and Analysis on Wiener Space.* W. de Gruyter, Berlin, 1991.

[24] H. Brezis. *Analyse Fonctionnelle. Théorie et Applications.* Masson, Paris, 1983.

[25] V. I. Burenkov. *Sobolev Spaces on Domains.* Teubner-Texte zur Mathematik, Vol.137, Stuttgart, 1998.

[26] M. Campiti and G. Metafune. Ventcel's boundary conditions and analytic semigroups. *Arch. Math.* 70 (1998), 377–390.

[27] M. Campiti, G. Metafune and D. Pallara. Degenerate self-adjoint evolution equations on the unit interval. *Semigroup Forum* 57 (1998), 1–36.

[28] Z. Q. Chen. Pseudo Jordan domains and reflecting Brownian motions. *Probab. Theory Relat. Fields* 94 (1992), 271–280.

[29] Z. Q. Chen. On reflecting diffusion processes and Skorokhod decompositions. *Probab. Theory Relat. Fields* 94 (1993), 281–315.

[30] Ph. Clément and C. A. Timmermans. On C_0-semigroups generated by differential operators satisfying Ventcel's boundary conditions. *Indag. Math.* 89 (1986), 379–387.

[31] D. Cohn. *Measure Theory.* Birkhäuser, Basel, 1980.

[32] G. Dal Maso and U. Mosco. Wiener's criterion and Γ-convergence. *Appl. Math. Option* 15 (1987), 15–63.

[33] D. Daners. Elliptic and parabolic equations and domain perturbation. *Brigham Young University, Provo, Utah,* 1999.

[34] D. Daners. Robin boundary value problems on arbitrary domains. *Trans. Amer. Math. Soc.* 352 (2000), 4207–4236.

[35] R. Dautray and J. L. Lions. *Analyse Mathématique et Calcul Numérique Vol. 3.* Masson, Paris, 1987.

[36] R. Dautray and J. L. Lions. *Analyse Mathématique et Calcul Numérique Vol. 8.* Masson, Paris, 1987.

[37] R. Dautray and J. L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 1. Physical Origins and Classical Methods.* Springer-Verlag, Berlin, 1988.

[38] R. Dautray and J. L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 2. Functional and Variational Methods.* Springer-Verlag, Berlin, 1988.

[39] E. B. Davies. *Heat kernels and Spectral Theory.* Cambridge University Press, Cambridge, 1989.

[40] E. B. Davies. Two-dimensional Riemannian manifolds with fractal boundaries. *J. London Math. Soc.* 49 (1994), 343–356.

[41] W. D. Edmunds and W. D. Evans. *Spectral Theory and Differential Operators.* Oxford Science Publications, Oxford, 1989.

[42] K. J. Engel and R. Nagel. *One-parameter Semigroups for Linear Evolution Equations.* Springer-Verlag, Berlin, 2000.

[43] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC. Press, Boca Raton, Florida, 1992.

[44] W. D. Evans and D. J. Harris. Sobolev embeddings for generalized ridged domains. *Proc. London Math. Soc.* 54 (1987), 141–175.

[45] W. D. Evans and D. J. Harris. On the approximation numbers of Sobolev embeddings for irregular domains. *Quart. J. Math. Oxford* 40 (1989), 13–42.

[46] K. J. Falconer. *Fractal Geometry.* Mathematical Foundations and Applications, Jonh Wiley, 1990.

[47] A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli. C_0-semigroups generated by second order differential operators with general Wentzell boundary conditions. *Proc. Amer. Math. Soc.* 128 (2000), 1981–1989.

[48] A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli. The heat equation with generalized Wentzell boundary conditions. *To appear in J.E.E.*

[49] A. Favini and S. Romanelli. Analytic semigroups on $C[0,1]$ generated by some classes of second order differential operators. *Semigroups Forum* 56 (1998), 362–372.

[50] H. Federer. *Geometric Measure Theory.* Springer-Verlag, Berlin, 1969.

[51] M. Fukushima. A construction of reflecting barrier Brownion motions for bounded domains. *Osaka J. Math.* 4 (1967), 183–215.

[52] M. Fukushima. Regular representation of Dirichlet spaces. *Trans. Amer. Math. Soc.* 155 (1971), 455–473.

[53] M. Fukushima. *Dirichlet Forms and Symmetric Markov Processes.* Amsterdam, North-Holland, 1980.

[54] M. Fukushima and Y. Lejan. On quasi-support of smooth measures and closability of pre-Dirichlet forms. *Osaka J. Math.* 28 (1991), 837–845.

[55] M. Fukushima, Y. Oshima and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes.* Amsterdam, North-Holland, 1994.

[56] M. Fukushima, K. I. Sato and S. Taniguchi. On the closable parts of pre-Dirichlet forms and the fine support of underlying measures. *Osaka J. Math.* 28 (1991), 517–535.

[57] M. Fukushima and M. Tomisaki. Reflecting diffusions on Lipschitz domains with cusps: Analytic construction and Skorohod representation. *Potential Anal.* 4 (1995), 377–408.

[58] M. Fukushima and M. Tomisaki. Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps. *Probab. Theory Relat. Fields* 106 (1996), 521–557.

[59] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin, 1986.

[60] G. Greiner. Perturbing the boundary conditions of a generator. *Houston J. Math.* 13 (1987), 213–229.

[61] P. Grisvard. *Singularities in Boundary Value Problems.* Springer-Verlag, Berlin, 1992.

[62] P. Hajlasz and O. Martio. Traces of Sobolev functions on fractal type sets and characterization of extension domains. *J. Funct. Anal.* 143 (1987), 221–246.

[63] P. W. Jones. Quasiconformal mappings and extendabibity of functions in Sobolev spaces. *Acta Math.* 147 (1981), 71–88.

[64] S. Karrmann. Gaussian estimates for second order operators with unbounded coefficients. J. Math. Anal. Appl. 258 (2001), 320–348.

[65] T. Kato. *Perturbation Theory for Linear Operators.* Springer-Verlag, Berlin, 1984.

[66] P. C. Kunstmann. L^p-spectral properties of the Neumann Laplacian on horns, comets and stars. *Math. Z.* 242 (2002), 183–201.

[67] P. C. Kunstmann and H. Vogt. Weighted norm estimates and L_p-spectral independence of linear operators. *Preprint*, 2002.

[68] J. L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications. Vol. I.* Springer-Verlag, Berlin, 1972.

[69] Z. M. Ma and M. Röckner. *Introduction to the Theory of Non-Symmetric Dirichlet Forms.* Springer-Verlag, Berlin, 1992.

[70] J. Maly and W. P. Ziemer. *Fine Reguralіty of Solutions of Elliptic Partial Differential Equations.* Providence, Amer. Math. Soc., 1997.

[71] P. Mandl. *Analytical Treatment of One-Dimensional Markov Processes.* Springer-Verlag, Berlin, 1968.

[72] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability.* Cambridge University Press, Cambridge, 1995.

[73] V. G. Maz'ja. *Sobolev Spaces.* Springer-Verlag, Berlin, 1985.

[74] V. G. Maz'ya and Y. Netrusov. Some conterexamples for the theory of Sobolev spaces on bad domains. *Potential Anal.* 4 (1995), 47–65.

[75] V. G. Maz'ya and S. V. Poborchi. *Differentiable Functions on Bad Domains.* World Scientific, Singapore, 1997.

[76] G. Metafune. Analyticity for some degenerate one-dimensional evolution equations. *Studia Math.* 127 (1998), 251–276.

[77] M. K. V. Murthy and G. Stampacchia. Boundary value problems for some degenerate elliptic operators. *Ann. Mat. Pura Appl.* 80 (1968), 1–122.

[78] R. Nagel (ed.). *One-parameter Semigroups of Positive Operators.* Lect. Notes in Math. Vol. 1184, Springer-Verlag, Berlin, 1986.

[79] E. M. Ouhabaz. Propriétés d'ordre et de contractivité des semi-groupes avec applications aux opérateurs elliptiques. *Ph.D. Dissertation, Université de Franche-Comté, Besançon*, 1992.

[80] E. M. Ouhabaz. L^∞-contractivity of semigroups generated by sectorial forms. *J. London Math. Soc.* 46 (1992), 529–542.

[81] E. M. Ouhabaz. Gaussian estimates and holomorphy of semigroups. *Proc. Amer. Math. Soc.* 123 (1995), 1465–1474.

[82] E. M. Ouhabaz. Invariance of closed convex sets and domination criteria for semigroups. *Potential Anal.* 5 (1996), 611–625.

[83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, Berlin, 1983.

[84] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. Vol. I. Functional Analysis.* Academic Press, New York, 1980.

[85] W. Rudin. *Real and Complex Analysis.* McGraw-Hill, Inc., 1966.

[86] H. H. Schaefer. *Banach Lattices and Positive Operators.* Springer-Verlag, Berlin, 1974.

[87] E. M. Stein. *Singular Integral and Differential Properties of Functions.* Princeton University Press, New Jersey, 1970.

[88] P. Stollmann. Closed ideals in Dirichlet spaces. *Potential Anal.* 2 (1993), 263–268.

[89] P. Stollmann and J. Voigt. Perturbation of Dirichlet forms by measures. *Potential Anal.* 5 (1996), 109–138.

[90] A. D. Venttsel. On boundary conditions for multidimensional diffusion processes. *Theo. Prob. Appl.* 9 (1959), 164–177.

[91] M. Warma. Wenzell-Robin boundary conditions on $C[0,1]$. *Semigroup Forum* 65 (2002), 1–9.

Index

Zusammenfassung

Gegenstand dieser Dissertation ist das Studium des Laplaceoperators mit Robin-, Neumann- und Wentzell-Robinrandbedingungen. Via quadratische Formen ist es einfach, Realisierungen des Laplaceoperators mit Dirichlet- oder Neumannrandbedingungen auf den Räumen $L^2(\Omega)$ zu definieren, wobei Ω eine offene Menge im $\mathbb{R}^N$ ist. Hierbei betrachten wir allgemeine Robinrandbedingungen; d.h. Randbedingungen gegeben durch Maße.

Im ersten Teil dieser Arbeit definieren wir einen neuen Begriff von Kapazität; die relative Kapazität. Die relative Kapazität ist immer kleiner als die klassische Kapazität. Relativ polare Mengen im Inneren von offenen Mengen sind polare Mengen. Wenn die offene Menge nicht regulär ist, dann existieren relativ polare Teilmengen des Randes mit positiver Kapazität. Wir illustrieren diese Aussage mit Beispielen.

Ein positives Borelmaß μ auf dem Rand heißt zulässig, falls relativ polare Mengen des Randes μ-Nullmengen sind. Für ein zulässiges Borelmaß μ auf dem Rand der offenen Menge Ω definieren wir eine positive, bilineare, symmetrische und abschließbare Form auf $L^2(\Omega)$. Der Operator, der mit dem Abschluß der Form assoziiert ist, ist eine Realisierung des Laplaceoperators und erzeugt eine holomorphe, stark stetige Halbgruppe. Diese liegt zwischen den vom Dirichlet und Neumann Laplaceoperator erzeugten Halbgruppen. Außerdem ist unter Regularitäts- und Lokalitätsbedingungen jede symmetrische und stark stetige Halbgruppe auf $L^2(\Omega)$, welche zwischen Dirichlet und Neumann Laplace Halbgruppen liegt, durch ein zulässiges Borelmaß gegeben.

Ist $\mu = \sigma$ das $(N-1)$-dimensionale Hausdorffmaß, dann ist der assoziierte Operator der Laplaceoperator mit den klassischen Robinrandbedingungen. Wir zeigen mit Beispielen, daß σ im allgemeinen nicht zulässig ist, aber es existiert ein Teilmenge S des Randes der offenen Menge, so daß man immer Robinrandbedingungen auf S und Dirichletrandbedingungen auf dem Komplement hat. Der Robin Laplaceoperator erzeugt eine stark stetige Halbgruppe auf $L^2(\Omega)$, die eine Gaußsche Abschätzung mit modifizierten Exponenten erfüllt. Außerdem ist das Spektrum des Robin Laplaceoperators auf $L^p(\Omega)$ unabhängig von $p \in [1, \infty)$.

Ist $\mu = 0$, so ist der assoziierte Operator der Neumann Laplaceoperator. Wir beweisen, daß die Neumann Laplace Halbgruppe aus Kernoperatoren besteht, jedoch ist der Kern singulär.

Im zweiten Teil der Arbeit beweisen wir, daß für beschränkte Lipschitz-Gebiete die Lösung des inhomogenen Robin Problems hölderstetig bis auf den Rand ist. Mit Hilfe dieser Resultate zeigen wir, daß der Teil des Robin Laplaceoperators auf den Räumen der stetigen Funktionen eine holomorphe, stark stetige Halb-

gruppe erzeugt. Am Schluß beweisen wir, daß der Laplaceoperator mit Wentzell-Robinrandbedingungen eine holomorphe, stark stetige Halbgruppe auf dem Raum der stetigen Funktionen auf einem abgeschlossenen Intervall erzeugt.

Erklärung:

Hiermit erkläre ich, daß ich die Arbeit selbständig und nur mit den angegebenen Hilfsmitteln angefertigt habe. Alle Stellen, die anderen Werken entnommen sind, wurden durch Angabe der Quellen kenntlich gemacht.

Ulm, den 13. Mai 2002.

Lebenslauf

Name	Mahamadi Warma	
Geboren	01. Januar 1967	
Geburtsort	Koumbri (Burkina Faso)	
Staatsbürgerschaft	Burkinisch	
Anschrift	Heidenheimer Str.82/502 89075 Ulm Tel.: 0731-9215249	
beruflich	Angewandte Analysis Universität Ulm 89069 Ulm Tel.: 0731-5023591	
Anstellungen	Seit 1999	Tätigkeit als wissenschafliche Hilfskraft an der Universität Ulm
	02/1998-07/1998	Stipendium des DAAD an der Universität Ulm
Studium	Seit 1998	Promotionsstudium in Mathematik an der Universität Ulm
	12/1995- 12/1997	Fortbildung an der Universität Porto-Novo (Benin)
	10/1988-12/1994	Studium in Mathematik (DEA) an der Universität Ouagadougou (Burkina Faso)
	12/1994	DEA de Mathématiques (Diplom)

	12/1993	Maitrise de Mathématiques
	12/1991	Licence de Mathématiques
Schulbildung	1981-1988	Lycée Yadéga de Ouahigouya (Burkina Faso)
	07/1988	Baccalaureat (Abitur)
	1975/1981	Ecole Primaire et Publique de Koumbri (Burkina Faso)