Math 685.
Lecture 5.

Methods for least squares approximation:

1) Normal eqns

$$A^TAx = A^Tb \leftarrow \text{fastest}$$

$$y = c_0 + c_1x + \ldots + c_nx^n$$

$$y_1 = c_0 + c_1x_1 + \ldots + c_nx_1^n$$

$$\vdots$$

$$y_m = c_0 + c_1x_m + \ldots + c_nx_m^n$$

x	$x_1 \ldots \ldots x_m$
y	$y_1 \ldots y_m$

$y = f(x)$ - least squares approximant

$$A = \begin{bmatrix} 1 & x_1 & \ldots & x_1^n \\ \vdots & & & \\ 1 & x_m & \ldots & x_m^n \end{bmatrix} \quad b = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

if $(A^TA)^{-1}$ exists (full rank problem $Ax=b$ with no rank deficiency)

then normal eqns provide unique sol.

→ conditioning problem

⇒ non-accurate answers

2) Augmented system

$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \qquad \begin{aligned} I \cdot r + Ax &= b \\ A^T r &= 0 \end{aligned}$$

"+" more robust, not as sensitive

"−" storage requirements are high

3) <u>Transformation</u>:

have to preserve the norm

$$\|P(Ax-b)\|_2 = \|Ax-b\|_2 = \|r\|_2$$

$\Rightarrow P$ has to be orthogonal : $P^TP = I$.

(A) QR - decomposition : $A = Q \cdot R \leftarrow \begin{pmatrix} \nabla \\ 0 \end{pmatrix}$

orthogonal

$$\|r\|_2^2 = \|b - Ax\|_2^2 = \|b - QRx\|_2^2 =$$

$$\|QQ^Tb - QRx\|_2 = \|\underbrace{Q^Tb}_{c} - Rx\|_2^2 = \|c - Rx\|^2$$

$R_1 x = c_1$
reduced form

$c = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$ $R = \begin{bmatrix} \nabla \\ 0 \end{bmatrix} \rightarrow R_1$

Since $\|Qy\| = \|y\|$

$$A = Q\begin{bmatrix} R \\ 0 \end{bmatrix} = (Q_1\ Q_2)\begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1 R$$

reduced form

rank$(A) = n \Rightarrow$ full rank, unique sol

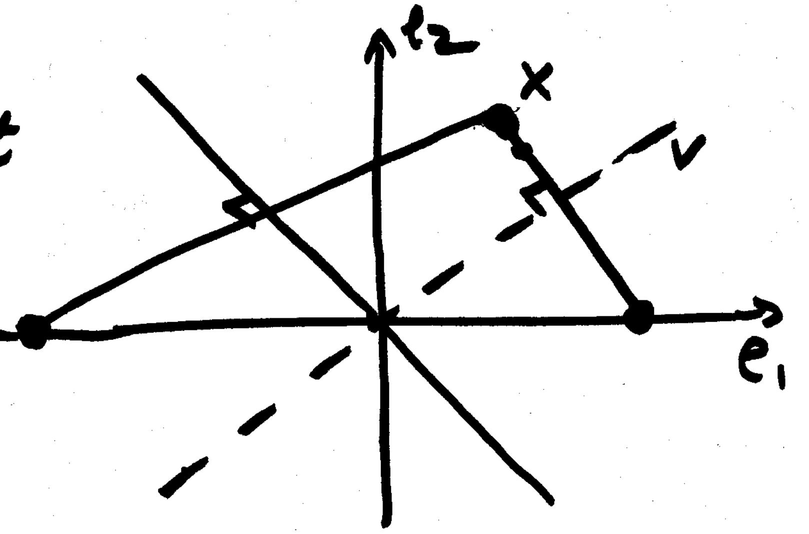
rank$(A) < n \Rightarrow$ rank-deficiency
R becomes singular.

$\Rightarrow$ column-pivoting is required.

Methods for finding Q:

1) Householder
2) Givens
3) Gram - Schmidt

$$H = I - 2\frac{v \cdot v^T}{v^T \cdot v}$$



$$\|Hv\| = \|v\|$$

$$Hv = \alpha \cdot e_1$$

H reflects in $span(A)^T$ so that all or some of vector components become zero, except for pivoting pivoting.

$$v = sign(x_1) \cdot \|x\| e_1 + x$$

then $\quad Fx = \pm\|x\| \cdot e_1$

Ex.

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{H} \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

$$Ha = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \cdot e_1 \leftarrow e_1 = \text{first column of } I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$\left(I - 2\frac{v \cdot v^T}{v^T v}\right) \cdot a = \alpha \cdot e_1$$

$$a - 2\vec{v} \cdot \underbrace{\frac{v^T a}{v^T v}}_{const} = \alpha e_1 \quad \Big| \times \left(\frac{v^T v}{v^T a}\right)$$

$$\underbrace{\frac{v^T v}{2 v^T a}}_{\text{multiple}} \cdot (a - \alpha e_1) = \vec{v}$$

multiple ⇒ Choose $\vec{v} = a - \alpha e_1$

since $\|v\|_2 = \|\alpha e_1\| = |\alpha|$

$\alpha = -\operatorname{sign}(a_1) \cdot \|a\|_2$

$$\boxed{H = I - 2\frac{v \cdot v^T}{v^T \cdot v}}$$ with $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-\alpha \\ 1 \\ 1 \end{pmatrix}$

$\|a\| = \sqrt{3}$ $\qquad \alpha = -\sqrt{3}$ $\qquad = \begin{pmatrix} 1+\sqrt{3} \\ 1 \\ 1 \end{pmatrix}$

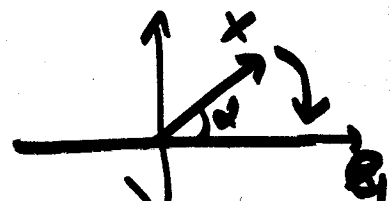
2) Givens rotation.

$\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ $\qquad c = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}$ $\qquad s = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}$

$\tan \alpha = \frac{s}{c} = \frac{a_2}{a_1}$

↖ angle of rotation

$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{G} \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}$



(1) rotate 1 & 2 $\qquad G_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$G_1 v = \begin{pmatrix} \alpha_1 \\ 0 \\ 1 \end{pmatrix}$

(2) rotate 1 & 3 $\qquad G_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

$G_2 G_1 v = \begin{pmatrix} \alpha_2 \\ 0 \\ 0 \end{pmatrix}$

$\|G_2 G_1 v\| = \|v\|$