Math 685/CSI 700/OR 682 Take-home Exam
given 03/22/10, due in class 04/05/10

Problem 1. Show how to evaluate the following expressions in a numerically stable fashion:

(a) $\dfrac{1}{1+2x} - \dfrac{1-x}{1+x}$, for $|x| \ll 1$, (b) $\sqrt{x+\dfrac{1}{x}} - \sqrt{x-\dfrac{1}{x}}$, for $x \gg 1$

Problem 2. Consider the problem of obtaining the scalar $f(x) = x_1 - x_2$ from a vector $(x_1, x_2) \in \mathbb{C}^2$. Find the condition number of this problem using infinity norm. When do you expect the problem to become ill-conditioned?

Problem 3. Prove that if Gaussian elimination with partial pivoting is applied to a matrix A that is diagonally dominant by columns, then no row interchanges will occur.

Problem 4.

$$A = \begin{pmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0.217 \\ 0.254 \end{pmatrix}.$$

The exact solution of $Ax = b$ is $x = (1, -1)^T$. Further, let two approximate solutions $x_1 = (0.999, -1.001)^T$, $x_2 = (0.341, -0.087)^T$ be given.
(a) Does the more accurate solution have a smaller residual?
(b) Determine the exact A^{-1} and $\texttt{cond}(A)$ with respect to the maximum norm.
(c) Explain the discrepancy observed in (a) by linking the residuals to the corresponding backward error.

Problem 5.
$C = (A^T A)^{-1}$ with $\texttt{rank}(A) = n$ is frequently used in statistics, where it is called a covariance matrix. Suppose we have obtained a decomposition $A = QR$.
(a) Prove that $C = (R^T R)^{-1}$.
(b) Design an algorithm for computing the diagonal entries of C requiring $n^3/3$ or less floating point operations.
Hint: Use the special structure of C and look for optimal algorithms in this context.

Problem 6. EXTRA CREDIT.
Let $A^T Ax = A^T b$, $(A^T A + F)\tilde{x} = A^T b$, $2||F||_2 \le \sigma_n(A)^2$, where $\sigma_n(A)$ denotes the smallest singular value of A. Show that if $r = b - Ax, \tilde{r} = b - A\tilde{x}$, then $\tilde{r} - r = (I - A(A^T A + F)^{-1} A^T)Ax$ and consequently

$$||\tilde{r} - r||_2 \le 2\texttt{cond}(A)\frac{||F||_2}{||A||_2}||x||_2.$$

What does this estimate accomplish? Hint: Notice the role of $(A^T A)^{-1} F$ in the above estimate and use the relationship $||(A^T A)^{-1}||_2 \cdot ||A||_2^2 = \texttt{cond}(A)^2$. The following fact from matrix algebra might be useful: for any M s.t. $||M|| < 1$, $||(I + M)^{-1}|| \le \dfrac{1}{1 - ||M||}$.