

Math 685/CSI 700/OR 682 Take-home Final Exam
given 04/26/10, due 05/12/10

Problem 1. Consider the problem of evaluating the function $f(x) = e^x$ for large positive x . Give the conditioning of this problem in absolute and relative sense and explain why the problem is ill-conditioned or well-conditioned.

Problem 2. Find an orthogonal matrix Q and a value α such that

$$Q \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

Problem 3. Let A be a singular upper Hessenberg matrix having no zero entries on its subdiagonal. Show that QR iteration applied to A converges to an exact eigenvalue in only 1 step. This is the reason why convergence of shifted QR method is accelerated when the shift is approximately equal to an eigenvalue.

Problem 4. Analyze the convergence of the fixed point method $x_{k+1} = f(x_k)$ for computing zeros $\alpha_1 = -1, \alpha_2 = 2$ of the function $f(x) = x^2 - x - 2$ when the following iteration functions are used: (a) $f_1(x) = x^2 - 2$, (b) $f_2 = \sqrt{2+x}$, (c) $f_3 = -\sqrt{2+x}$ and (d) $f_4 = 1 + 2/x, x \neq 0$.

Problem 5. Use the linear ODE $x' = \lambda x$ to analyze accuracy and stability of the Heun's method $x_{k+1} = x_k + \frac{h_k}{2}(k_1 + k_2)$, where $k_1 = f(t_k, x_k), k_2 = f(t_k + h_k, y_k + h_k k_1)$. You should show that the method is 2nd order accurate and characterize its stability region.

Problem 6.

Suppose that the altitude of the trajectory of a projectile is described by the 2nd order ODE $y'' = -4$. Suppose that the projectile is fired from position $t = 0$ and height $u(0) = 1$ and is to strike a target at position $t = 1$, also of height $u(1) = 1$.

(a) Solve this BVP by the shooting method:

Use the trapezoid rule with step $h = 1$ to derive a system of 2 equations to determine the initial slope at $t = 0$ required to hit the desired target at $t = 1$ ($s_0 = u'(0)$) and the final slope $s_1 = u'(1)$. Using the initial slope found above and a step size $h = 0.5$, use the trapezoid rule once again to estimate the projectile height at $t = 0.5$

(b) Solve the same BVP using a finite difference method with $h = 0.5$. Compare the height estimation at $t = 0.5$ with that obtained via shooting method.

(c) Solve the same BVP using collocation method at the point $t = 0.5$, together with the boundary values, to determine a quadratic polynomial $u(t)$ approximating the solution. Again, compare the height prediction at the point $t = 0.5$ with previous methods.

Problem 7. Extra credit.

(1) Solve Problem 6 using Galerkin method.

(2) Implement each of the above methods numerically for the BVP $u'' = -(1 + e^u)$ with the same boundary conditions as above. Comment on accuracy and stability of each method.