Math 678.
Lecture 7.

Poisson problem in $\mathbb{R}^n_+$ – half-space:

$$\mathbb{R}^n_+ = \{(x_1, \dots x_n) / x_n > 0\}$$

$$\underbrace{\varphi^x(y) = \Phi(y - \tilde{x})},\ \text{satisfies} \begin{cases} \Delta\varphi^x = 0, & \mathbb{R}^n_+ \\ \varphi^x = \Phi(y-x), & \partial\mathbb{R}^n_+ \end{cases}$$

if $y \in \partial\mathbb{R}^n_+ \Rightarrow y_n = 0 \Rightarrow$

$$|y - x| = |(y_1 - x_1, \dots, 0 - x_n)| = |(y_1 - x_1, \dots\ 0 + x_n)|$$
$$= |y - \tilde{x}|$$

We showed: $\displaystyle u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,dy =$

$$= \int_{\partial\mathbb{R}^n_+} K(x,y) g(y)\,dy$$

$K(x,y)$ – Poisson kernel

We need: $\displaystyle \lim_{\substack{x \to x^0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x^0) \quad \forall x^0 \in \partial\mathbb{R}^n_+$

$$|u(x) - g(x^0)| \le \underbrace{\int_{\partial\mathbb{R}^+_n \cap B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\,dy}_{\le\varepsilon}$$
$$+ \underbrace{\int_{\partial\mathbb{R}^+_n \setminus B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\,dy}_{\checkmark}$$

if $|x - x^0| \le \frac{\delta}{2}$, $|y - x^0| > \delta$

$$|y - x^0| \le |y - x| + |x - x^0| \le |y - x| + \tfrac{1}{2}|y - x^0|$$
$$\Rightarrow |y - x| \ge \tfrac{1}{2}|y - x^0|$$
$$\frac{1}{|y-x|} \le \frac{2}{|y - x_0|}$$

$$G_{y_i} = \Phi_{y_i}(y-x) - \Phi_{y_i}(|x|(y-x^*))$$

$$\Phi_{y_i}(y-x) = \frac{1}{n\alpha(n)} \frac{x_i - y_i}{|x-y|^n}$$

$\forall y \in \partial B(0,1)$:

$$\Phi_{y_i}(|x|(y-x^*)) = -\frac{1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{(|x||y-x^*|)^n}$$

$$\frac{\partial G}{\partial \nu}(x,y) = \sum_{i=1}^{n} y_i\, G_{y_i}(x,y) =$$

$$= -\frac{1}{n\alpha(n)} \frac{1}{|x-y|^n} \sum_{i=1}^{n} y_i((y_i - x_i) - y_i|x|^2 + x_i)$$

$$= -\frac{1}{n\alpha(n)} \frac{1-|x|^2}{|x-y|^n}$$

$$\left[u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, dS(y) \right]$$ representation formula for $B(0,1)$

if radius is r instead of 1,
$\tilde{u}(x) = u(rx)$ solves BVP on $B(0,1)$ with
$\tilde{g}(x) = g(rx)$ at $\partial B(0,1)$

$\Rightarrow$ in $B(0,r)$:

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \cdot \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y), \quad x \in B(0,r)$$

Alternative derivation for $B_2(0,r)$:

$$\Omega = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 < a^2\}$$

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ u = h(\theta) & (x,y) \in \partial\Omega \end{cases}$$

$$u = u(r,\theta), \quad \begin{array}{l} r = r(x,y) \\ \theta = \theta(x,y) \end{array}$$

$$J = \int_{\partial \mathbb{R}^+_n \setminus B(x^0,\varepsilon)} K(x,y)\,|g(y)-g(x^0)|\,dy \le$$

$$\le \|g\|_{L^\infty} \int \frac{|y-x^0|\cdot 2x_n}{n\alpha(n)\,|x-y|^n}\,dy \le$$

$$\le \frac{2x_n\cdot \|g\|_{L^\infty}\cdot 2^n}{n\,\alpha(n)} \int \frac{1}{|x_0-y|^{n-1}}\,dy \longrightarrow 0 \quad \text{as } x_n \to 0+.$$

$$|u(x)-g(x^0)| \le \varepsilon + o(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0 \quad \text{for } |x-x^0| \text{ suff. small}$$

<u>Example 1</u> $\mathbb{R}^2_+ = \{(x_1,x_2)\in\mathbb{R}^2 \mid x_2>0\}$

Find $\varphi^x(y)$ s.t. $\begin{cases} \Delta\varphi^x(y)=0, & y\in\mathbb{R}^2_+ \\ \varphi^x(y)=\Phi(y-x), & y\in\partial\mathbb{R}^2_+ \end{cases}$

We want to fix $x\in\mathbb{R}^2_+$ and then find a point z s.t. $\Phi(y-z)=\Phi(y-x)$ $\forall y\in\partial\Omega$.

$\Phi(y-x)$ is harmonic as long as $y\neq x$.

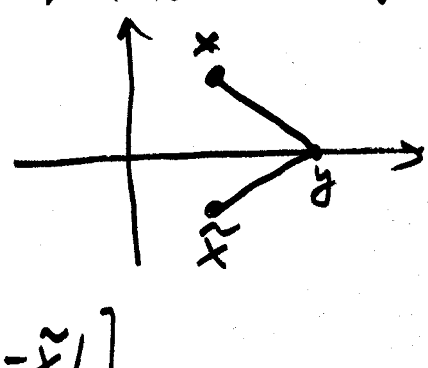
Notice that $\Phi(y-z)$ is harmonic for all $z\notin\mathbb{R}^2_+\equiv\Omega$

So $z=z(x)$ s.t. $\begin{cases} z\notin\mathbb{R}^2_+ \text{ and} \\ \Phi(y-z)=\Phi(y-x) \text{ on } \partial\Omega \end{cases}$

$n=2$ $\quad \Phi(y-z) = -\frac{1}{2\pi}\ln|y-z|$

$\forall y\in\partial\mathbb{R}^2_+$, $|y-x| = |(y_1,0)-(x_1,x_2)| = |(y_1,0)-(x_1,-x_2)| = |y-\tilde{x}|$

$\Rightarrow z(x)=\tilde{x}$ - reflection in $\partial\Omega$



$\Rightarrow \varphi^x(y) = \Phi(y-\tilde{x}) = -\frac{1}{2\pi}\ln|y-\tilde{x}|$

$$G(x,y) = \Phi(y-x)-\Phi(y-\tilde{x})$$
$$= -\frac{1}{2\pi}\left[\ln|y-x| - \ln|y-\tilde{x}|\right]$$

Example. $\Omega = B_2(0,1) \subset \mathbb{R}^2$

$$" \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$$

Fix $x \in B_2(0,1)$. $\Phi(y-x) = -\frac{1}{2\pi} \ln|y-x|$

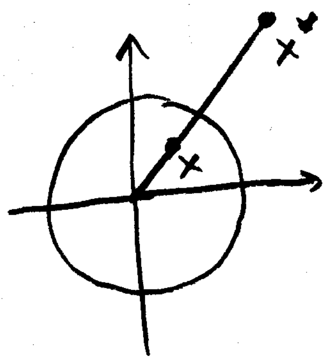
Need: $\varphi^x(y)$ s.t. $\begin{cases} \Delta \varphi^x(y) = 0 & \text{in } \Omega \\ \varphi^x(y) = \Phi(y-x) & \text{on } \partial\Omega \end{cases}$

If $y \in \partial\Omega = \partial B_2(0,1)$ $|y| = 1$

$$|y-x|^2 = \underbrace{|y|^2}_{=1} - 2y\cdot x + |x|^2 =$$

$$= |x|^2 - 2y\cdot x + 1 =$$

$$= |x|^2|y|^2 - 2y\cdot x + 1 =$$



$$= |x|^2\left(|y|^2 - 2y\cdot\frac{x}{|x|^2} + \frac{1}{|x|^2}\right)$$

$$= |x|^2\Big(\underbrace{|y|^2 - 2y\cdot\frac{x}{|x|^2} + \frac{|x|^2}{|x|^4}}_{|y-x^*|^2,\quad x^* = \frac{x}{|x|^2}}\Big)$$

$$|y-x|^2 = |x|^2 \cdot |y-x^*|^2$$

$\varphi^x(y) = \Phi(|x|(y-x^*))$ — corrector fct for $B_2(0,1)$.

$$\Rightarrow G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*))$$

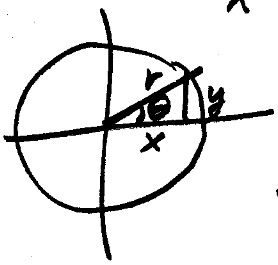
$$= -\frac{1}{2\pi}\left(\ln|y-x| - \ln\left(|x|\cdot|y-x^*|\right)\right)$$

In n-dimensional case,

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*))$$

Let's show that this $G(x,y)$ allows to solve the BVP $\begin{cases} \Delta u = 0 & \text{in } B(0,1) \\ u = g & \text{on } \partial B(0,1) \end{cases}$

$$u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial G}{\partial \nu}(x,y)\, dS$$



$$x^2+y^2=r^2 \Rightarrow 2x = 2r\cdot r_x$$

$$r_x = \frac{x}{r} = \cos\theta$$

$$r_y = \frac{y}{r} = \sin\theta$$

$$\tan\theta = \frac{y}{x} \Rightarrow$$

$$\frac{1}{\cos^2\theta}\cdot\theta_x = -\frac{y}{x^2} \Rightarrow \theta_x = -\frac{\sin\theta}{r}$$

$$\frac{y}{x^2} = \frac{\sin\theta}{\cos\theta\cdot x} \Rightarrow \theta_x = -\frac{\sin\theta\cdot\cos^2\theta}{\cancel{\cos\theta}\cdot x}$$

Same way $\theta_y = \frac{\cos\theta}{r}$

$$\Rightarrow \cancel{u_{xx}}\ \frac{\partial u}{\partial x}(r,\theta) = u_r\cdot r_x + u_\theta\cdot\theta_x =$$

$$= u_r\cdot\cos\theta + u_\theta\cdot\left(-\frac{\sin\theta}{r}\right)$$

$$\Rightarrow \frac{\partial}{\partial x} = \cos\theta\cdot\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}$$

Also $\frac{\partial}{\partial y} = \sin\theta\cdot\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial\theta}$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \cos^2\theta\frac{\partial^2}{\partial r^2} + \cancel{2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}} - \cancel{2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}}$$

$$+ \frac{\sin^2\theta}{r}\cdot\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2} - \cancel{2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}} + \cancel{2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\Rightarrow \boxed{u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0}$$ Laplace in polar coords.

Separation of variables:

$$u(r,\theta) = R(r)\cdot\Theta(\theta)$$

$$\Rightarrow R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\Rightarrow \frac{\Theta''}{\Theta} = -\frac{r^2R''}{R} - \frac{rR'}{R} = -\lambda \leftarrow \text{const}$$

I. $\begin{cases} \Theta'' = -\lambda\Theta & 0 < \theta < 2\pi \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$

ODE

$\Rightarrow \begin{cases} \Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta) \\ \lambda_n = n^2 \end{cases}$

II. $r^2 R_n'' + r R_n' = \lambda_n R_n$

ODE $r^2 R_n'' + r R_n' - n^2 R_n = 0$

Look $R_n = r^\alpha \Rightarrow (\alpha^2 - n^2) r^\alpha = 0$

$\Rightarrow \underline{R_n(r) = r^n \text{ or } r^{-n}}, \; n \geq 1$

when $n = 0 \Rightarrow R_0(r) = 1$

$r^2 R'' + r R' = 0 \Rightarrow R_1(r) = \ln r$

$\Rightarrow u_n(r,\theta) = R_n(r) \cdot \Theta(\theta) =$

$= \begin{cases} (C_n r^n + D_n \cdot \frac{1}{r^n})(A_n \cos n\theta + B_n \sin n\theta), & n \geq 1 \\ A_0 [C_0 + D_0 \ln r], & n = 0 \end{cases}$

since $u(r=a, \theta) = h(\theta)$

$\Rightarrow A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\theta) \cos n\theta \, d\theta$

$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\theta) \sin n\theta \, d\theta$

$$u(r,\theta) = \sum_{n=0}^{\infty} u_n(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a^2 - r^2}{a^2 - 2ar \cdot \cos(\theta - \varphi) + r^2} \right) h(\varphi) \, d\varphi$$

Rectangular coordinates: $x \in \Omega$

$x' \in \partial\Omega$

$|x - x'|^2 = a^2 - 2a \cdot r \cos(\theta - \varphi) + r^2$ by Cosine Thm.

$$\Rightarrow u(r,\theta) = \frac{1}{2\pi} \int_{|x'|=a} \frac{u(x') \cdot (a^2 - |x'|^2)}{|x - x'|^2} \, ds$$