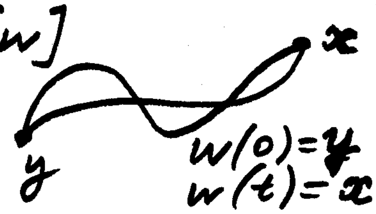
Math 678.
Lecture 25.

$$\begin{cases} u_t + H(Du, x) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

CE: $\begin{cases} \dot{p}(s) = -D_x H \\ \dot{z}(s) = D_p H \cdot p - H \\ \dot{x}(s) = D_p H \end{cases} \Rightarrow$ Hamilton eqn: $\begin{cases} \dot{x}(s) = D_p H \\ \dot{p}(s) = -D_x H \end{cases}$

$$I[w(\cdot)] := \int_0^t L(\dot{w}(s), w(s))\, ds$$

Variational formulation: $\min_{w \in \mathcal{A}} I[w]$



Thm: If $X = \operatorname{argmin}_{w \in \mathcal{A}} I[w]$ then

it satisfies Euler-Lagrange eqns:

$$-\frac{d}{ds}\left(D_v L(\dot{x}(s), x(s))\right) + D_x L(\dot{x}(s), x(s)) = 0 \qquad 0 \le s \le t$$

Proof:

$y : [0,t] \to \mathbb{R}^n \quad y(0) = y(t) = 0$ smooth curve

$W(\cdot) = x(\cdot) + \tau y(\cdot), \quad \tau \in \mathbb{R}$

$W(0) = x(0) + \tau y(0) = y + \tau \cdot 0 = y$

$w(t) = x(t) + \tau y(t) = x + \tau \cdot 0 = x$ $\Rightarrow w \in \mathcal{A}$

$\Rightarrow I[x] \le I[w]$ since $X = \operatorname{argmin}_{w \in \mathcal{A}} I[w]$

$u(\tau) := I[x(\cdot) + \tau y(\cdot)]$ has a min at $\tau = 0$

$\Rightarrow u'(\tau) = 0$ when $\tau = 0$. $\quad u(\tau) = \int_0^t L(\dot{x}(s) + \tau \dot{y}(s), x(s) + \tau y(s))\, ds$

$$u'(\tau) = \int_0^t \sum_{i=1}^n \left(L_{v_i} \cdot \dot{y}_i(s) + L_{x_i} \cdot y_i\right) ds$$

$$= \sum_{i=1}^n \int_0^t \left(-\frac{d}{ds}(L_{v_i}) + L_{x_i}\right) y_i\, ds$$

$$0 = u'(0) = \sum_{i=1}^n \int_0^t \underbrace{\left[-\frac{d}{ds} L_{v_i}(\dot{x}(s), x(s)) + L_{x_i}(\dot{x}(s), x(s))\right]}_{\overset{!}{0}} y_i(s)\, ds \Rightarrow \text{EL eqns.}$$

EL eqns characterise crit. pts, so not only minima of $I[w]$ satisfy these eqns. This is a system of n 2nd order ODEs.

$$\underset{\text{EL system}}{\begin{pmatrix}\text{2nd order ODE} \\ \text{system} \\ n \text{ eqns}\end{pmatrix}} \rightsquigarrow \underset{\text{Hamilton system}}{\begin{pmatrix}\text{1st order ODE} \\ \text{system} \\ 2n \text{ eqns}\end{pmatrix}}$$

~~Chan~~ Introduce variables $p(s) := D_v L(\underset{= v}{\dot{x}(s)}, x(s))$

<u>Assume</u>: this makes a solvable eqn in v

$$p = D_v L(v, x) \Rightarrow v = V(p, x) \quad \forall p, x \in \mathbb{R}^n$$

<u>Def</u>. $H(p,x) := p \cdot V(p,x) - L(V(p,x), x)$

Hamiltonian

In case of $L = \frac{1}{2} m |v|^2 - \varphi(x)$

$$p = L_v = \cancel{\tfrac{1}{2}} m v$$

$$v = \frac{p}{m} = V(p, x)$$

$$\Rightarrow H = p \cdot \frac{p}{m} - \frac{1}{2} m |v|^2 + \varphi(x) = \underbrace{\frac{1}{2m} \cdot |p|^2}_{\text{kinetic}} + \underbrace{\varphi(x)}_{\text{potential}}$$

<u>Derivation of Hamilton eqns:</u>

We know: $p = D_v L$

$$\dot{x}(s) = v(p(s), x(s))$$

$$\begin{cases} H_{x_i} = \sum_{k=1}^{n} p_k V^k_{x_i}(p,x) - L_{v_k} \cdot V^k_{x_i} - L_{x_i} \\ \quad = - L_{x_i}(v, x) \\ H_{p_i}(p,x) = v^i(p,x) + \cancel{\sum_{k=1}^{n} \left(p_k V^k_{p_i} - L_{v_k} \cdot V^k_{p_i}\right)} \\ \quad = v^i(p,x) \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} H_{p_i}(p,x) = \dot{x}_i(s) \\ H_{x_i}(p,x) = -\dot{p}_i(s) \end{cases}}$$ Hamilton equations 1st order $2n$ eqns

$$\frac{d}{ds} H(p,x) = \sum_{i=1}^{n} \left(H_{p_i} \cdot \dot{p}_i + H_{x_i} \cdot \dot{x}_i\right) = \sum \left(-H_{p_i} H_{x_i} + H_{x_i} \cdot H_{p_i}\right) = 0$$

$$H = \text{const}$$

From now on: $H = H(p)$ so $H_{x_i} = 0$.

$$\left.\begin{array}{l}(1)\ L: \mathbb{R}^n \to \mathbb{R} \text{ convex} \\ (2)\ \lim\limits_{v \to \infty} \frac{L(v)}{|v|} = +\infty\end{array}\right\} \Rightarrow L^*(p) := \sup_{v \in \mathbb{R}^n} \{p \cdot v - L(v)\} \quad p \in \mathbb{R}^n$$

Legendre transform

<u>Thm</u>: (Convex duality of Hamiltonian & Lagrangian)

If L satisfies (1)-(2), take $H = L^*$ then

(1) $H(p)$ is convex and $\lim\limits_{|p| \to \infty} \frac{H(p)}{|p|} = \infty$

(2) $L = H^*$

<u>Corollary</u>: ~~TF~~ TFAE:
$$\begin{cases} p \cdot v = L(v) + H(p) \\ p = DL(v) \\ v = DH(p) \end{cases}$$

Go back to Hamilton-Jacobi: $\begin{cases} u_t + H(Du) = 0 \\ u = g \end{cases}$

CE: $\begin{cases} \dot{p} = 0 \\ \dot{x} = DH(p) \\ \dot{z} = DH(p) \cdot p - H(p) \end{cases}$ in case $H = H(p)$

When smooth soln exists (locally) we have

$$\dot{z} = \underbrace{DH(p) \cdot p - H(p)}_{L} = \underbrace{\dot{x}}_{v} \cdot p - H(p) = L(\dot{x}) \Rightarrow$$

$$u(x,t) = \int_0^t L(\dot{x}(s))\, ds + g(x(0))$$

Wanted: modification of this formula that works even when soln becomes non-smooth.

Change action functional: $\min\limits_{\substack{w \\ w(t)=x}} \int_0^t L(\dot{w}(s))\, ds + \underbrace{g(w(0))}_{\text{initial data}}$

<u>Define</u>: $u(x,t) := \inf\limits_{w \in C^1} \left\{ \int_0^t L(\dot{w}(s))\, ds + g(w(0)) \,\middle|\, w(t) = x \right\}$

<u>Hopf-Lax formula</u>: $u(x,t) = \min\limits_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\}$