Math 678.
Lecture 19.

Solitons:

$u_t + 6u \cdot u_x + u_{xxx} = 0$ KdV eqn

$u(x,t) = v(\underbrace{x - \sigma t}_{s})$ σ - velocity of traveling wave

$-\sigma v' + 6v \cdot v' + v''' = 0$

$-\sigma v + 3v^2 + v'' = A = \text{const}$ (1)

$-\sigma v \cdot v' + 3v^2 v' + v'' v' = A v'$

$v'' v' = A v' + \sigma v \cdot v' - 3v^2 v'$

$\left(\frac{(v')^2}{2}\right)' = (Av)' + \left(\frac{\sigma v^2}{2}\right)' - (v^3)'$

$\frac{(v')^2}{2} = Av + \frac{\sigma v^2}{2} - v^3 + B$ (2)

Take $s \to \pm\infty \Rightarrow$ we are interested in v s.t.
$v, v', v'' \to 0$

$A = 0$ from (1)
$B = 0$ from (2)

$\Rightarrow \quad \frac{(v')^2}{2} = v^2\left(\frac{\sigma}{2} - v\right)$

$(v')^2 = v^2(\sigma - 2v)$

$v' = \pm v\sqrt{\sigma - 2v}$ pick $\ominus$

$\int \frac{dv}{v\sqrt{\sigma - 2v}} = -\int ds$

$v(s) = \frac{\sigma}{2}\operatorname{sech}^2\left(\frac{\sqrt{\sigma}}{2}(s - c)\right)$

$u(x,t) = \frac{\sigma}{2}\operatorname{sech}^2\left(\frac{\sqrt{\sigma}}{2}(x - \sigma t - c)\right)$ – soliton wave Soln to KdV eqn.

Transform methods.

1. Fourier transform.

<u>Def.</u> $u \in L^1(\mathbb{R}^n)$

direct $\hat{u}(y) = \mathcal{F}(u) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)\,dx$, $y \in \mathbb{R}^n$

inverse $\check{u}(y) = \mathcal{F}^{-1}(u) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} u(x)\,dx$ $y \in \mathbb{R}^n$

<u>Plancherel's Thm:</u>

$u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \Rightarrow \hat{u}, \check{u} \in L^2(\mathbb{R}^n)$ and

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$$

Consider $\{u_k\}_{k=1}^{\infty} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and

$u_k \to u$ in $L^2(\mathbb{R}^n)$

$\|\hat{u}_k - \hat{u}_j\|_{L^2(\mathbb{R}^n)} = \|\widehat{(u_k - u_j)}\|_{L^2(\mathbb{R}^n)} = \|u_k - u_j\|_{L^2(\mathbb{R}^n)}$

$\Rightarrow \{\hat{u}_k\}_{k=1}^{\infty}$ - Cauchy sequence $\Rightarrow$ converges to $\hat{u}$.

$\hat{u}$ - is called a Fourier transform of u

$\hat{u}_k \to \hat{u}$ in $L^2(\mathbb{R}^n)$.

does not depend on the choice of $\{u_k\}$

<u>Properties.</u>

1) $\widehat{D^\alpha u} = (iy)^\alpha \hat{u}$, α - multi-index

2) $\int_{\mathbb{R}^n} u \cdot \bar{v}\,dx = \int_{\mathbb{R}^n} \hat{u} \cdot \bar{\hat{v}}\,dy$

3) $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \Rightarrow \widehat{(u * v)} = (2\pi)^{n/2}\, \hat{u} \cdot \hat{v}$

4) $u = (\hat{u})^\vee = \mathcal{F}^{-1}(\mathcal{F}(u))$

Examples.

1) Heat eqn IVP : $\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$\hat{u} = \mathcal{F}_x(u) = \hat{u}(y)$ $\quad u(x,t)$

$y \in \mathbb{R}^n$

$\mathcal{F}(u_t - \Delta u) = 0$ $\qquad \hat{u} = e^{-t|y|^2}\hat{g}$

$$\begin{cases} \hat{u}_t + |y|^2 \hat{u} = 0 & t>0 \\ \hat{u} = \hat{g} & t=0 \end{cases}$$

$$u(x,t) = \mathcal{F}^{-1}(\underbrace{e^{-t|y|^2}}_{\hat{F} = e^{-t|y|^2}}\hat{g}) = \mathcal{F}^{-1}(\hat{F}\cdot\hat{g}) = \frac{1}{(2\pi)^{n/2}}(F*g)$$

$$\mathcal{F}(F*g) = (2\pi)^{n/2}\hat{F}\cdot\hat{g}$$

$$\frac{1}{(2\pi)^{n/2}}(F*g) = \mathcal{F}^{-1}(\hat{F}\cdot\hat{g})$$

$$F = \mathcal{F}^{-1}(e^{-t|y|^2}) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-t|y|^2}e^{ix\cdot y}\,dy =$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-t|y|^2 + ix\cdot y}\,dy \; (=)$$

$$\int_{-\infty}^{+\infty} e^{ia\cdot x - bx^2}\,dx = \int_{-\infty}^{+\infty}\left(e^{-\frac{a^2}{4b}}\right)e^{-u^2}\,dx = e^{-\frac{a^2}{4b}}\cdot\sqrt{\frac{\pi}{b}}$$

$$ia\cdot x - bx^2 = -\frac{a^2}{4b} - \left(\sqrt{b}\,x - \frac{a}{2\sqrt{b}}i\right)^2$$

$$bx^2 - ia\cdot x + \frac{a^2}{4b}$$

$$u = \sqrt{b}\,x - \frac{a}{2\sqrt{b}}i \qquad \begin{matrix} b = t \\ a = x_i \end{matrix}$$

$$(=)\; \frac{1}{(2\pi)^{n/2}}\cdot\prod_{j=1}^{n}\left(\int_{-\infty}^{+\infty} e^{-ty_i^2 + ix_i\cdot y_i}\,dy_i\right) = \frac{1}{(2\pi)^{\frac{n}{2}}}\prod_{j=1}^{n}\left(e^{-\frac{x_i^2}{4t}}\sqrt{\frac{\pi}{t}}\right)$$

$$= \frac{1}{(2t)^{n/2}}\cdot e^{-|x|^2/4t} \Rightarrow \boxed{u(x,t) = \frac{1}{(4\pi t)^{n/2}}\int e^{-\frac{|x-y|^2}{4t}}g(y)\,dy}$$

$$\left(\frac{\pi}{t}\right)^{n/2}\cdot e^{-\frac{|x|^2}{4t}}$$