# Math 678.
## Lecture 17.

In the proof last time

$$u(x,t) = \underbrace{\frac{1}{\gamma_{n+1}}\,\frac{2\alpha(n)}{(n+1)\alpha(n+1)}}_{\overset{?}{=}\,\frac{1}{\gamma_n}} \left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{g(y)\,dy}{\sqrt{t^2-|y-x|^2}}\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{h(y)\,dy}{\sqrt{t^2-|y-x|^2}}\right)\right]$$

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} \leftarrow \text{volume of } B_n(0,1)$$

$$\gamma_{n+1} = 1\cdot 3\cdot 5 \cdots (n-1) = (n-1)!!$$

$$\gamma_n = 1\cdot 2\cdot 4 \cdots n = n!!$$

$$\frac{1}{\gamma_{n+1}}\,\frac{2\alpha(n)}{(n+1)\alpha(n+1)} = \frac{2\cdot \pi^{n/2}\cdot \Gamma\left(\frac{n+3}{2}\right)}{(n+1)!!\,\pi^{\frac{n+1}{2}}\cdot\Gamma\left(\frac{n+2}{2}\right)} = \frac{2\,\Gamma\left(\frac{n+3}{2}\right)}{\sqrt{\pi}\,(n+1)!!\,\Gamma\left(\frac{n+2}{2}\right)}$$

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$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

Properties: $\Gamma(z+1) = z\Gamma(z)$, $\quad \Gamma(n) = (n-1)!$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\cdot\sqrt{\pi}$$

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$n = 2k \qquad \frac{n+3}{2} = (k+1)+\frac{1}{2}$, $\quad n+1 = 2k+1$, $\quad 2(k+1)-1 = 2k+1$

$\frac{n+2}{2} = k+1$, $\quad \Gamma\left(\frac{n+3}{2}\right) =$

$$\Rightarrow \frac{2}{\cancel{\sqrt{\pi}}\,\cancel{(2k+1)!!}}\cdot\frac{\cancel{(2k+1)!!}\cdot\cancel{\sqrt{\pi}}}{2^{k+1}\cdot k!} = \frac{1}{2^k\cdot k!} = \frac{1}{(2k)!!} = \boxed{\frac{1}{\gamma_n}}$$

**Nonhomogeneous Wave Eqn:**

$$\begin{cases} u_{tt} - \Delta u = f & \mathbb{R}^n\times(0,\infty) \\ u = 0 & \\ u_t = 0 & t = 0 \end{cases}$$

## Duhamel's principle:

$$u(x,t) = \int_0^t u(x,t;s)\,ds \quad \text{where}$$

$$\begin{cases} u_{tt}(\cdot;s) - \Delta u(\cdot;s) = 0 & \mathbb{R}^n \times (s,\infty) \\ \left.\begin{aligned} u(\cdot,s) &= 0 \\ u_t(\cdot,s) &= f(\cdot,s) \end{aligned}\right\} & \{t=s\} \end{cases}$$

Justification:

$$u_t(x,t) = \underbrace{u(x,t;t)}_{=0} + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)$$

$$u_{tt}(x,t) = u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds =$$

$$= f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds$$

$$\Delta u(x,t) = \int_0^t \Delta(u,t;s)\,ds = \int_0^t u_{tt}(u,t;s)\,ds$$

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = f(x,t) & t>0 \\ u(x,0) = u_t(x,0) = 0 & x \in \mathbb{R}^n \end{cases}$$

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$$\begin{bmatrix} \text{Soln of} \\ u_{tt} - \Delta u = f \\ u = g \\ u_t = h \end{bmatrix} \text{on } \{t=0\} = \begin{bmatrix} \text{Soln of} \\ u_{tt} - \Delta u = f \\ u = 0 \\ u_t = 0 \end{bmatrix} + \begin{bmatrix} \text{Soln of} \\ u_{tt} - \Delta u = 0 \\ u = g \\ u_t = h \end{bmatrix}$$

Examples:

$\boxed{n=1}$ $\begin{cases} u_{tt} - \Delta u = f \\ u = 0 \\ u_t = 0 \end{cases}$ d'Alembert formula:

$$u(x,t;s) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(y,s)\,dy$$

$$\begin{cases} u_{tt}(\cdot;s) - \Delta u(\cdot;s) = 0 \\ u(\cdot;s) = 0 \\ u_t(\cdot;s) = f(\cdot;s) \end{cases}$$

$$u(x,t) = \int_0^t u(x,t;s)\,ds = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\,dy\,ds =$$

$$= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds = \frac{1}{2}\int_0^t \int_{x-s'}^{x+s'} f(y, t-s')\,dy\,ds'$$

$s = t-s'$ $\quad s' = t-s$ $\quad ds' = -ds$ $\quad 0 \le s \le t$

$x - s' \le y \le x + s'$ $\quad t \le s' \le 0$

$\boxed{n=3}$ $\quad \begin{cases} u_{tt} - \Delta u = 0 \\ \left.\begin{array}{l} u(\cdot, s) = 0 \\ u_t(\cdot, s) = f \end{array}\right\} (t=s) \end{cases}$

Kirchhoff's formula: $\quad u(x,t;s) = (t-s) ⨍_{\partial B(x,t-s)} f(y,s)\,dS(y)$

$$u(x,t) = \int_0^t (t-s) ⨍_{\partial B(x,t-s)} \left(f(y,s)\,dS\right) ds = \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)\,dS}{t-s}\,ds =$$

from Surface area $\quad r = t-s$

$$= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\,dS\,dr = \frac{1}{4\pi}\int_{B(x,r)} \frac{f(y, t-|y-x|)}{|y-x|}\,dy$$

Energy methods:

Uniqueness Proof:

$$(*) \begin{cases} u_{tt} - \Delta u = f & U_T \\ u = g & \Gamma_T \\ u_t = h & U \times \{t=0\} \end{cases}$$

To show: this BIVP has at most one solution in $C^2(U_T)$.

Suppose $\tilde{u}$ solves $(*)$. Then $w := u - \tilde{u}$

$$\Rightarrow \begin{cases} w_{tt} - \Delta w = 0 \\ w = 0 \\ w_t = 0 \end{cases}$$

$$e(t) := \frac{1}{2}\int_U (w_t^2 + |Dw|^2)\,dx \qquad 0 \le t \le T$$

energy

$$\dot{e}(t) = \int_U (w_t w_{tt} + Dw \cdot Dw_t)\,dx =$$

$$= \int_U (w_t w_{tt} - \Delta w \cdot w_t)\,dx = \int_U w_t (\underbrace{w_{tt} - \Delta w}_{=0})\,dx$$

on $\partial U$, $\begin{matrix} w = 0 \\ w_t = 0 \end{matrix}$ on $\partial U \times [0,T]$

For all $0 \le t \le T$, $e(t) = e(0) = 0$

In particular $w_t = 0$
$Dw = 0$

Since $w = 0$ on $U \times \{t=0\} \Rightarrow w \equiv 0$ on $U_T$.

<u>Domain of dependence</u> (Finite propagation speed):

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

fix $x_0 \in \mathbb{R}^n$, $t > 0$

If $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$ then
$u \equiv 0$ within the cone $C = \{(x,t) \mid 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}$

<u>Proof</u>: $e(t)$ as above: $e(t) = \frac{1}{2}\int_{U = B(x_0, t_0 - t)} (u_t^2 + |Du|^2)\,dx$

$$\dot{e}(t) = \int_{B(x_0,t_0-t)} (u_t \cdot u_{tt} + Du \cdot Du_t)\,dx =$$

$$= \int_{B(x_0,t_0-t)} (\underbrace{u_t \cdot u_{tt} - \Delta u \cdot u_t}_{=0 \ (\text{wave eqn})\cdot u_t})\,dx + \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} \cdot u_t\,dS -$$

$$- \frac{1}{2}\int_{\partial B(x_0,t_0-t)} (u_t^2 + |Du|^2)\,dx$$

$$= \int_{\partial B(x_0,t_0-t)} \left(\frac{\partial u}{\partial \nu} u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2\right)dx$$

$$\left| \int_{\partial B(x_0, t_0 - t)} \frac{\partial u}{\partial \nu} u_t \right| \le \left( \int u_t^2 \right)^{1/2} \cdot \left( \int |Du|^2 \right)^{1/2} \le$$

$$\left| \int f \cdot g \right| \le \left( \int f^2 \right)^{1/2} \cdot \left( \int g^2 \right)^{1/2}$$ Cauchy-Schwartz inequality

$$2 a \cdot b \le a^2 + b^2$$

$$\le \frac{1}{2} \int_{\partial B(x_0, t_0 - t)} u_t^2 + \frac{1}{2} \int_{\partial B(x_0, t_0 - t)} |Du|^2$$

$$\Rightarrow \int_{\partial B(x_0, t_0 - t)} \left( \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |Du|^2 \right) dx \le 0$$

$$\Rightarrow \dot{e}(t) \le 0 \Rightarrow e(t) \le e(0) = 0 \Rightarrow e(t) \equiv 0$$

$\Rightarrow e \equiv 0$ on cone $C$

$$\begin{cases} u_t \equiv 0 \\ Du \equiv 0 \end{cases} \Rightarrow u \equiv 0 \text{ on cone } C.$$