Math 678.
Lecture 15.

Def. $D_{U\times\{t_0\}}$ - domain of influence of $U\times\{t_0\}=$

$$=\{(x,t)\in\mathbb{R}^n\times[0,\infty)\mid u(x,t)\neq 0\}$$

u ↑ solution of the wave equation vanishing outside of $U\times\{t_0\}$

$D_{(x_0,t_0)}$ - domain of influence of a point (x_0,t_0)

$$=\Big\{(x,t)\in\bigcap_{\varepsilon>0} D_{B(x_0,\varepsilon)\times\{t_0\}}\Big\}$$

Domain of dependence of (x_0,t_0) is the set of all points (x,t) s.t. $D_{(x,t)}$ includes (x_0,t_0).

$\boxed{n=1}$

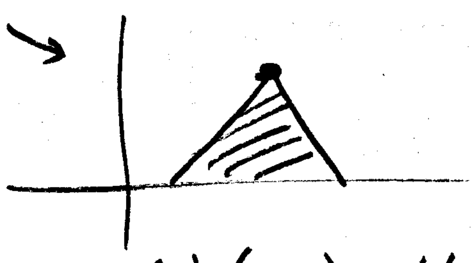
$$u(x,t)=\frac{1}{2}\big(g(x+t)+g(x-t)\big)+\frac{1}{2}\int_{x-t}^{x+t}h(y)\,dy$$

forward cone → Domain of influence of $x_0=\{(x,t): x\in(x_0-t,\,x_0+t)\}$

Domain of dependence of $(x_0,t_0)=$

$$=\{(x,t): x\in[x_0-(t_0-t),\,x_0+(t_0-t)]\}$$

↗ backward cone →



$\boxed{n=3}$

$$u(x,t)=⨍_{\partial B(x,t)}\big(g(y)+\nabla g(y)\cdot(y-x)+t h(y)\big)\,dS(y)$$

Domain of influence of a ball $B(x_0,r)=$

$$=\{(x,t)\mid r-t<|x-x_0|<r+t\}$$

Domain of dependence of $(x_0,t_0)=\{(x,t)\mid |x-x_0|=t_0-t\}$

2d. Method of Descent (Hadamard Descent)

$u(x_1, x_2, t)$ – solution to $\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^2 \times (0,\infty) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$

$$\begin{cases} \bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t) \\ \bar{g}(x_1, x_2, x_3) = g(x_1, x_2) \\ \bar{h}(x_1, x_2, x_3) = h(x_1, x_2) \end{cases}$$

$\bar{u}$ solves a 3-d wave equation with

$$\bar{u}(\cdot, 0) = \bar{g}$$
$$\bar{u}_t(\cdot, 0) = \bar{h}$$

$\vec{x} = (x_1, x_2) \Rightarrow \vec{y} = (y_1, y_2) \Rightarrow$

$$\bar{u}(x, x_3, t) = \fint_{\partial B_3(x, x_3, t)} \{\bar{g}(y, y_3) + \nabla\bar{g}(y, y_3)\cdot[(y, y_3) - (x, x_3)] + t\bar{h}(y, y_3)\}\, dS_3(y, y_3)$$

We may set $x_3 = 0$ (since nothing depends on it)

$\partial B_3(x, 0, t)$ = union of 2 graphs over $B_2(x, t)$ given by $y_3 = \pm\sqrt{t^2 - y_1^2 - y_2^2}$

$$\int_{\partial\Omega} f\, dS = \int_{\Omega} \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

$$dS_3 = \sqrt{1 + \partial_{y_1}^2(y_3) + \partial_{y_2}^2(y_3)} = \sqrt{1 + \frac{y_1^2}{t^2 - |y|^2} + \frac{y_2^2}{t^2 - |y|^2}}$$

$$= \sqrt{\frac{t^2 - |y|^2 + |y|^2}{t^2 - |y|^2}} = \frac{t}{\sqrt{t^2 - |y|^2}}$$

$$\nabla_3 \bar{g}(y, y_3) = (\partial_{y_1}(g), \partial_{y_2}(g), 0)$$

$$\Rightarrow \quad \nabla\bar{g}(y, y_3)\cdot[(y, y_3) - (x, x_3)] = \nabla_2 g \cdot (y - x)$$

$$\Rightarrow u(x_1, x_2, t) = \frac{2}{t^2 \cdot \gamma} \int_{B_2(x,t)} \left(g(y) + \nabla g(y) \cdot (y-x) + t h(y)\right) \frac{t}{\sqrt{t^2 - |y|^2}} \, dy$$

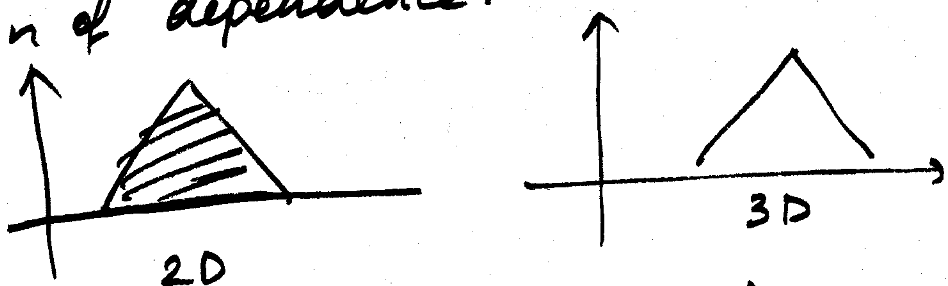
$$\gamma = |\partial B_3(0,1)| = 4\pi$$

$$|B_2(0,1)| = \pi \qquad \frac{2}{4\pi t^2} = \frac{1}{2} \cdot \frac{1}{|B_2(x,t)|}$$

$$|B_2(x,t)| = \pi \cdot t^2$$

$$\Rightarrow u(x_1, x_2, t) = \frac{1}{2} ⨍_{B_2(x,t)} \left[g(y) + \nabla g(y) \cdot (y-x) + t h(y)\right] \cdot \frac{t \, dy}{\sqrt{t^2 - |y|^2}}$$

Domain of dependence:



Solution in odd dimensions ($n \geq 3$)

Recall: ① $\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ U = G & \text{on } \mathbb{R}_+ \times \{t=0\} \\ U_t = H & \end{cases}$

(EPD eqn for spherical means)

② $\tilde{U} = rU$, $\tilde{G} = rG$, $\tilde{H} = rH$

$\Rightarrow \tilde{U}$ solves 1d wave eqn $\begin{cases} \tilde{U}_{rr} = \tilde{U}_{tt} \\ \left.\begin{array}{l} \tilde{U} = \tilde{G} \\ \tilde{U}_t = \tilde{H} \end{array}\right\} t=0 \\ \tilde{U} = 0, \ r=0 \end{cases}$

③ In 3d we used

$$\lim_{r \to 0} \frac{\tilde{U}}{r} = u(x,t)$$

General case: $n = 2k+1$, odd $\left(k = \frac{n-1}{2}\right)$

Define U, G, H same as before

But now we will use

$$\tilde{U} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}}\left(r^{n-2} U\right)$$
$$\tilde{G} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}}\left(r^{n-2} G\right)$$
$$\tilde{H} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}}\left(r^{n-2} H\right)$$

Claim: 1) $\tilde{U} \in C^2([0,\infty)\times[0,\infty))$ and satisfies 1d wave equation

2) $\lim_{r\to 0} \frac{\tilde{U}}{\gamma_n r} = u(x,t)$, where $\gamma_n = 1\cdot 3 \cdot \ldots \cdot (n-2) = (n-2)!!$

If 1) holds, use d'Alembert result to get

$$\tilde{U}(x,r,t) = \frac{1}{2}\left(\tilde{G}(r+t) - G(t-r)\right) + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(s)\,ds \qquad 0 \le r \le t$$

Now use 2):

$$u(x,t) = \lim_{r\to 0}\frac{\tilde{U}}{\gamma_n r} =$$
$$= \frac{1}{\gamma_n}\left[\lim_{r\to 0}\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \lim_{r\to 0}\frac{1}{2r}\int_{t-r}^{t+r}\tilde{H}(s)\,ds\right]$$
$$= \frac{\tilde{G}'(t) + \tilde{H}(t)}{\gamma_n}$$

$$\Rightarrow u(x,t) = \frac{1}{\gamma_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2} \fint_{\partial B(x,t)} g\, dS\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} h\, dy\right)\right]$$