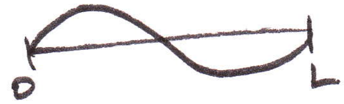
Math 678.
Lecture 13.

Wave equation.



$u_{tt} - u_{xx} = 0$ 1D

$u_{tt} - \Delta u = 0$ in n-D

$u_{tt} - \Delta u = f$ – non-hom. case

$u(x,t)$ - displacement

$\boxed{n=1}$

IVP in 1-d
$$\begin{cases} u_{tt} - u_{xx} = 0, & \mathbb{R}\times(0,\infty) \\ u = g & \\ u_t = h & , \text{ on } \mathbb{R}\times\{t=0\} \end{cases}$$

$$\cancel{u_t + u_x}\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u} = 0$$

$$v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u(x,t)$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0$$

$$\Rightarrow a(x) = v(x,0),\ b = 1$$

$$v(x,t) = a(x-t)$$

$$u_t - u_x = a(x-t)$$

$b = -1$

Transport eqn:
$$\begin{cases} u_t + b\cdot Du = 0, & \forall \\ u = g, & t=0 \end{cases}$$
$$u(x,t) = g(x - tb)$$

$$u_t + b\cdot Du = f$$
$$\Rightarrow u(x,t) = g(x-tb) + \int_0^t f(x + (s-t)b, s)\,ds$$

$$\Rightarrow u(x,t) = d(x+t) + \int_0^t a(x + (t-s) - s)\,ds$$

where $u(x,0) = d(x)$

From initial data: $u(x,0) = d(x) = g(x)$

$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x)$$

$$\Rightarrow u(x,t) = g(x+t) + \int_0^t h(x+t-2s)\,ds - \int_0^t g'(x+t-2s)\,ds$$

Change variables $\Rightarrow$ $y = x+t-2s$ $\Rightarrow$ $s=0 \Rightarrow y = x+t$

$dy = -2ds$ $\qquad$ $s = t \Rightarrow y = x-t$

$$u(x,t) = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy - \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t)$$

$$\boxed{u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy}$$

d'Alembert's formula

If $h \equiv 0$ $\quad u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right]$

traveling waves with speeds 1 and -1 resp.

<u>Nonhomogeneous case</u>:

$$\begin{cases} u_{tt} - u_{xx} = f & \mathbb{R}\times(0,\infty) \\ u(\cdot,0) = g & \text{on } \mathbb{R} \\ u_t(\cdot,0) = h & \end{cases}$$

<u>To show</u>: $u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+h} h(y)\,dy$

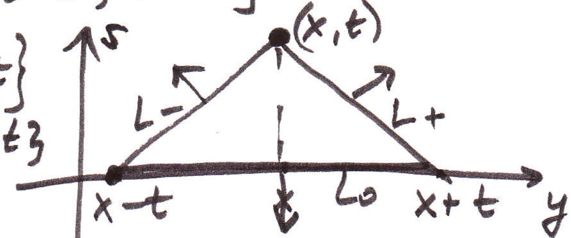
$$\boxed{+\frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f(y,\tau)\,dy\,d\tau}$$ the new part

<u>Solution</u>: $C_1(x,t) = \{(y,s) \mid |y-x| < t-s,\ s>0\}$

$L_+ = \{(y,s) \mid s = -y + x + t,\ 0<s<t\}$

$L_- = \{(y,s) \mid s = y - x + t,\ 0<s<t\}$

$L_0 = \{(y,0) \mid x-t < y < x+t\}$



Characteristic triangle

$$\int_{C_1(x,t)} f\,dy\,ds = \int_{C_1(x,t)} (u_{tt} - u_{xx})\,dy\,ds =$$

$$= \int_{\partial C_1} (u_t \nu_2 - u_x \nu_1)\,dl \;\ominus$$

$\nu = (\nu_1, \nu_2)$ – unit outward normal of $\partial C_1(x,t)$

$$\nu = \begin{cases} (0,-1) & \text{on } L_0 \\ \frac{(1,1)}{\sqrt{2}} & \text{on } L_+ \\ (-1,1)/\sqrt{2} & \text{on } L_- \end{cases}$$

$$\stackrel{\text{(}=\text{)}}{=} \int_{L_+} \frac{1}{\sqrt{2}}(u_t - u_x)\, dl + \int_{L_-} \frac{1}{\sqrt{2}}(u_t + u_x)\, dl - \int_{x-t}^{x+t} \underbrace{u_t(s,0)}_{\overset{\shortparallel}{h(s)}}\, ds$$

$\frac{\partial_t - \partial_x}{\sqrt{2}}$ – unit directional derivative with direction = integral direction on L_+

$\frac{\partial_t + \partial_x}{\sqrt{2}}$ – directional derivative on L_- direction – opposite to integral

$$\Rightarrow \int_{L_+} \frac{1}{\sqrt{2}}(u_t - u_x)\, dl = u(x,t) - \underbrace{u(x+t,0)}_{g(x+t)}$$

$$\int_{L_-} \frac{1}{\sqrt{2}}(u_t + u_x)\, dl = -\Big(\underbrace{u(x-t,0)}_{g(x-t)} - u(x,t)\Big)$$

$$\Rightarrow \int_{C_1(x,t)} f(y,s)\, dy\, ds = 2u(x,t) - g(x+t) - g(x-t) - \int_{x-t}^{x+t} h(s)\, ds$$

$$\Rightarrow u(x,t) = \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(s)\, ds + \underbrace{\frac{1}{2}\int_{C_1(x,t)} f(y,s)\, dy\, ds}_{}$$

$$\int_0^t \int_{x-(t-s)}^{x+t-s} f(y,s)\, dy\, ds$$