Wave equation.

\[ u_{tt} - u_{xx} = 0 \]
\[ u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \quad \text{for } n \geq 0 \]
\[ u_{tt} - \Delta u = f \quad \text{non-hom. case} \]
\[ u(x, t) \quad \text{displacement} \]

\( n = 1 \)

Initial Value Problem (IVP) in 1-D:

\[ \begin{cases} u_{tt} - u_{xx} = 0, & R \times (0, \infty) \\ u = g, & \text{on } R \times t = 0 \\ u_t = h, & \text{on } R \times t = 0 \end{cases} \]

\[ u_{tt} - u_{xx} = 0 \quad \Rightarrow \quad \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) u = 0 \]

\[ v(x, t) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) u(x, t) \]

\[ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0 \]

\[ \Rightarrow \quad a(x) = v(x, 0), \quad b = 1 \]

\[ v(x, t) = a(x-t) \]

\[ u_t - u_x = a(x-t) \]

\[ b = -1 \]

\[ \Rightarrow \quad u(x, t) = d(x+t) + \int a(x+(t-s)) \, ds \]

where \( u(x, 0) = d(x) \)

From initial data:

\[ u(x, 0) = d(x) = g(x) \]

\[ a(x) = v(x, 0) = u_t(x, 0) - u_x(x, 0) = h(x) - g'(x) \]

\[ \Rightarrow \quad u(x, t) = g(x+t) + \int h(x+t-2s) \, ds - \int g'(x+t-2s) \, ds \]

Change variables:

\[ \frac{dy}{ds} = -2 \quad \Rightarrow \quad y = x + t - 2s \]

\[ s = t \Rightarrow y = x + t \]
\[ u(x,t) = g(x+t) + \frac{1}{2} \int_{x-t}^{x+t} h(y) \, dy - \frac{1}{2} g(x+t) + \frac{1}{2} g(x-t) \]

\[ u(x,t) = \frac{1}{2} [g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) \, dy \]

**d'Alembert's formula**

If \( h = 0 \)
\[ u(x,t) = \frac{1}{2} [g(x+t) + g(x-t)] \]

Traveling waves with speeds 1 and -1 resp.

**Nonhomogeneous case:**
\[ \begin{cases} 
\partial_{tt} u - \partial_{xx} u = f & \text{in } \mathbb{R} \times (0,\infty) \\
\partial_x u(t,0) = g & \text{on } \mathbb{R} \\
\partial_x u(t,0) = h & \text{on } x = x = x + h
\end{cases} \]

To show: \[ u(x,t) = \frac{1}{2} [g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) \, dy \]

New part

**Solution:**
\[ C_1(x,t) = \{(y,s) \mid y-x<s<t, s>0\} \]

\[ L_+ = \{(y,s) \mid s = -y + x + t, 0 < s < t\} \]
\[ L_- = \{(y,s) \mid s = y - x + t, 0 < s < t\} \]
\[ L_0 = \{(y,0) \mid x-t < y < x+t\} \]

\[ \int_{C_1(x,t)} f \, dy \, ds = \int_{C_1(x,t)} (\partial_{tt} u - \partial_{xx} u) \, dy \, ds = 0 \]

**Boundary:**
\[ \partial_{C_1} = \{(0,1,0), (1,0,1), (0,1,1)\} - \text{normal of } \partial C_1(x,t) \]

\[ \nu = \{ (0,-1) \text{ on } L_0, (1,1) \text{ on } L_+, (-1,1)\sqrt{2} \text{ on } L_- \} \]
\[
\begin{aligned}
\sum \int_{L^+} \frac{1}{\sqrt{2}} (u_t - u_x) \, dl + \int_{L^-} \frac{1}{\sqrt{2}} (u_t + u_x) \, dl = \int_{x-t}^{x+t} u_t(s,0) \, ds
\end{aligned}
\]

- directional derivative with direction = integral direction on \( L^+ \)
- directional derivative on \( L^- \) - opposite to integral

\[
\begin{aligned}
\Rightarrow \int_{L^+} \frac{1}{\sqrt{2}} (u_t - u_x) \, dl &= u(x,t) - \frac{u(x,t,0)}{g(x+t)} \\
\int_{L^-} \frac{1}{\sqrt{2}} (u_t + u_x) \, dl &= -(u(x-t,0) - u(x,t)) / g(x-t)
\end{aligned}
\]

\[
\begin{aligned}
\int_{C_1(x,t)} f(y,s) \, dy \, ds &= 2u(x,t) - g(x+t) - g(x-t) \\
&\quad - \int_{x-t}^{x+t} h(s) \, ds
\end{aligned}
\]

\[
\begin{aligned}
\Rightarrow u(x,t) &= \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(s) \, ds + \frac{1}{2} \int_{x-t}^{x+t} f(y,s) \, dy \, ds
\end{aligned}
\]