Math 678.
Lecture 12.

Regularity for heat eqn.

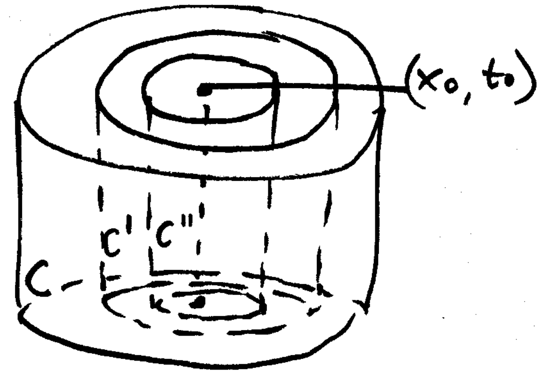
Thm. $u \in C_1^2(U_T)$ solves heat eqn in U_T
$\Rightarrow u \in C^\infty(U_T)$

Proof: $C(x_0, t_0; r) = \{(y,s) \mid |x_0 - y| \le r,\ t_0 - r^2 \le s \le t_0\}$
circular cylinder

$(x_0, t_0) \in U_T$, pick $r > 0$ s.t. $C(x_0, t_0; r) \subset U_T$.

$C' := C(x_0, t_0; \frac{3}{4} r)$

$C'' := C(x_0, t_0; \frac{1}{2} r)$



Cutoff $\xi = \xi(x,t)$ s.t. ξ-smooth

$0 \le \xi \le 1$

$\xi = 1$ on C'

$\xi \equiv 0$ near the boundary of C

Extend ξ ~~to~~ as $\xi \equiv 0$ on $\mathbb{R}^n \times [0, t_0] - C$

~~Suppose~~ Assume for the moment that $u \in C^\infty(U_T)$

$v(x,t) := \xi(x,t) u(x,t)$, $x \in \mathbb{R}^n$, $0 \le t \le t_0$.

$$-\begin{cases} v_t = \xi u_t + \xi_t u \\ \Delta v_x = \xi \Delta u + 2 D\xi \cdot Du + u \Delta \xi \end{cases} \qquad \left(\underbrace{u_t - \Delta u = 0}_{\text{heat eqn}}\right)$$

$$\Rightarrow \begin{cases} v_t - \Delta v = \xi_t u - 2 D\xi \cdot Du - u\Delta\xi := \tilde{f} \\ v = 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

$$\tilde{v} = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) \tilde{f}(y,s)\, dy\, ds \quad \text{- solves BVP}$$

$\Rightarrow v \equiv \tilde{v}$, by uniqueness

$$\Rightarrow v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) \tilde{f}(y,s)\, dy\, ds$$

Take $(x,t) \in C''$ $\quad \xi = 0$ outside, $\xi = 1$ on C'

$v_t - \Delta v = \tilde{f}$, $\quad v = \iint \Phi(x-y, t-s)\tilde{f}\, dy\, ds$.

$$\Rightarrow u(x,t) = \iint_C \Phi(x-y,t-s)\tilde{f}\, dy\, ds =$$

$$= \iint_C \Phi(x-y, t-s)\cdot\big[(\xi_s(y,s) - \Delta\xi(y,s))u(y,s) - 2D\xi(y,s)\cdot Du(y,s)\big]\, dy\, ds$$

$$= \iint_C \Phi(x-y,t-s)\big((\xi_s(y,s) - \Delta\xi(y,s)\big)u(y,s)\, dy\, ds + 2\iint_C D_y\Phi(x-y,t-s)\cdot D\xi(y,s)\cdot u(y,s)\, dy\, ds.$$

$$= \iint_C \Phi(x-y,t-s)\cdot\big[\xi_s - \Delta\xi + 2D_y\Phi\cdot D\xi\big]u(y,s)\, dy\, ds$$

$$= \iint_C K(x,t,y,s)\, u(y,s)\, dy\, ds$$

$$K(x,t,y,s) = \Phi(x-y,t-s)\cdot\big[\xi_s - \Delta\xi + 2D_y\Phi\cdot D\xi\big]$$

in C'. $\xi \equiv 1$, so $K(x,t,y,s) = 0$ $\forall (y,s) \in C'$

and $K(x,t,y,s)$ - smooth on $C \setminus C'$.

From here $u \in C^\infty$ in $C'' = C(x_0, t_0, \frac{1}{2}r)$

Same construction works for

$u^\varepsilon := \eta_\varepsilon * u$, where η_ε - mollifier

Then take $\varepsilon \to 0$ $\Rightarrow$ conclusion follows.

Estimate of derivatives

$$\max_{C(x,t;\frac{r}{2})} |D_x^k D_t^\ell u| \le \frac{C_{k\ell}}{r^{k+2\ell+n+2}} \|u\|_{L^1(C(x,t;r))}$$

for all $C(x,t;\frac{r}{2}) \subset C(x;t;r) \subset U_T$

for any u - soln of heat eqn.

Energy methods.

BVP ⊕ : $\begin{cases} u_t - \Delta u = f, & U_T \\ u = g, & \Gamma_T \end{cases}$ $U \subset \mathbb{R}^n$ - bdd open, $\partial U \in C^1$

T-termination time.

Thm. (Alternative proof of uniqueness).
$\exists$ at most one soln to ⊕ which is in $C_1^2(\bar{U}_T)$.

Proof. Suppose $\tilde{u}$ - second soln to ⊕.

$w := u - \tilde{u} \Rightarrow \begin{cases} w_t - \Delta w = 0, & U_T \\ w = 0; & \Gamma_T \end{cases}$

$$E(t) := \int_U w^2(x,t)\,dx, \quad 0 \le t \le T$$

$$\frac{dE}{dt} = \dot{E} = 2\int_U w \cdot w_t\,dx = 2\int_U w \cdot \Delta w\,dx = -2\int_U |Dw|^2 dx \le 0$$

$E(t) \le E(0) = 0 \quad 0 \le t \le T \Rightarrow E(t) \equiv 0 \Rightarrow w \equiv 0$ in U_T.

Backwards uniqueness:

$\begin{cases} u_t - \Delta u = 0 \\ u = g \text{ on } \partial U \times [0,T] \end{cases}$, $\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 \\ \tilde{u} = g \text{ on } \partial U \times [0,T] \end{cases}$

$u, \tilde{u}$ can be different at $t=0$.
If $u(x,T) = \tilde{u}(x,T)$, $x \in U \Rightarrow u \equiv \tilde{u}$ on U_T.

Proof:
1) $w := u - \tilde{u}$, $E(t) = \int_U w^2(x,t)\,dt$

$$\dot{E} = -2\int_U |Dw|^2 dx$$

$$\ddot{E} = -4\int_U Dw \cdot Dw_t\,dx = +4\int \Delta w \cdot \underset{\Delta w}{\underset{\parallel}{w_t}}\,dx$$

$$= 4\int_U (\Delta w)^2 dx$$

$w = 0$ on ∂V

$$\int_V |Dw|^2 dx = -\int_V w \cdot \Delta w \, dx \le \left(\int_V w^2 dx\right)^{1/2} \cdot \left(\int_V (\Delta w)^2 dx\right)^{1/2}$$

$$(\dot{E}(t))^2 = 4\left(\int_V |Dw|^2 dx\right)^2 \le \underbrace{\left(\int_V w^2 dx\right)}_{E}\underbrace{\left(\int_V 4(\Delta w)^2 dx\right)}_{\ddot{E}}$$

$$\Rightarrow \dot{E}^2 \le E \cdot \ddot{E} \Rightarrow E \cdot \ddot{E} - \dot{E}^2 \ge 0$$

Assume $E(t) > 0$ $\quad t_1 \le t < t_2$, $E(t_2) = 0$.

$[t_1, t_2] \in [0, T]$

If $f(t) = \log E(t)$

$$\dot{f} = \frac{\dot{E}}{E}, \quad \ddot{f} = \frac{\ddot{E}}{E} - \frac{(\dot{E})^2}{E^2} = \frac{\ddot{E} \cdot E - (\dot{E})^2}{E^2} \ge 0$$

f - ~~conca~~ convex

So $\forall \lambda \in (0,1)$, $t_1 < t < t_2$

$f((1-\lambda)t_1 + \lambda t) \le (1-\lambda) f(t_1) + \lambda f(t)$ by convexity

$f((1-\lambda)t_1 + \lambda t) \le (1-\lambda) f(t_1) + \lambda f(t)$

$\log E((1-\lambda)t_1 + \lambda t) \le (1-\lambda) \log E(t_1) + \lambda \log E(t)$

$$E((1-\lambda)t_1 + \lambda t) \le E(t_1)^{1-\lambda} \cdot E(t_2)^{\lambda}$$

$$0 < E((1-\lambda)t_1 + \lambda t_2) < E(t_1)^{1-\lambda} \cdot \underset{0}{\underset{\shortparallel}{E}}(t_2)^{\lambda} = 0.$$

$$\Rightarrow E(t) = 0 \text{ on } [t_1, t_2].$$

$$\Rightarrow \underline{E \equiv 0.}$$