Math 678. Homework 6 Solutions.

#1

$$x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, u(x_1, x_2, 0) = g(x_1, x_2)$$

First we identify $F(p, z, x) = x_1 p_1 + 2x_2 p_2 + p_3 - 3z = 0$. This gives $F_p = (x_1, 2x_2, 1), F_x = (p_1, 2p_2, 0), F_z = -3$. The CE system looks like

$$\begin{cases} \dot{x}(s) = F_p \\ \dot{z}(s) = F_p \cdot p \end{cases} = \begin{cases} \dot{x}_1(s) = x_1, \dot{x}_2(s) = 2x_2, \dot{x}_3(s) = 1 \\ \dot{z}(s) = x_1 p_1 + 2x_2 p_2 + p_3 = 3z \end{cases}$$

Initial conditions: $x(0) = (x_1^0, x_2^0, 0), z(0) = g(x_1^0, x_2^0)$. Integrating the CE system, we get

$$x_1 = x_1^0 e^s, \quad x_2 = x_2^0 e^{2s}, \quad x_3 = s, \quad z(s) = g(x_1^0, x_2^0)e^{3s}$$

Eliminating the initial conditions and s, we obtain

$$u(x, t) = z(s) = g(x_1 e^{-x_3}, x_2 e^{-2x_3})e^{3x_3}$$

#2

$$uu_{x_1} + u_{x_2} = 1, u(x_1, x_1) = x_1/2$$

Here $F(p, z, x) = zp_1 + p_2 - 1 = 0$ and $F_p = (z, 1), F_x = 0, F_z = p_1$. The CE system is thus:

$$\begin{cases} \dot{x}_1 = z \\ \dot{x}_2 = 1 \\ \dot{z} = zp_1 + p_2 = 1 \end{cases}$$

Imposing initial conditions $x(0) = (x_1^0, x_1^0), z(0) = x_1^0/2$, we get

$$x_2(s) = s + x_1^0, \quad z(s) = s + x_1^0/2, \quad x_1(s) = s^2/2 + (x_1^0/2)s + x_1^0$$

This implies $s = x_2 - x_1^0$ and $x_1^0 = \dfrac{2x_1 - x_2^2}{2 - x_2}$ and hence

$$u(x, t) = z(s) = x_2 - \frac{2x_1 - x_2^2}{2(2 - x_2)}$$

#3

$$\begin{cases} u_x + xu_t = 0 \\ u(x, 0) = 0 \\ u(0, t) = t \end{cases}$$

Applying Laplace transform in the time domain to both sides and denoting $\mathcal{L}(u) = F(s)$, we get

$$F_x(x, s) + x(sF(x, s) - F(x, 0)) = 0$$
$$F(x, 0) = 0, F(0, s) = \frac{1}{s^2}$$

Hence the equation becomes $F_x(x,s) = -xsF(x,s)$ which after integration gives $F(x,s) = C(s)e^{-sx^2/2}$ and since at $x = 0$ the value of $F(0,s) = \frac{1}{s^2} = C$, we get the formula $F(x,s) = \dfrac{e^{-sx^2/2}}{s^2}$. Using the table of Laplace transforms, we see that the solution can be computed as

$$u(x,t) = \Big(t - \frac{x^2}{2}\Big)u\Big(t - \frac{x^2}{2}\Big)$$

where $u(t - x^2/2)$ is a step function, equals to 1 as $t - x^2/2 > 0$ and zero otherwise.