Math 678. Homework 3 Solutions.

#1

We need to derive the formula for the solution of the IVP

$$\begin{aligned} u_t - \Delta u + cu &= f. \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u &= g, \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{aligned}$$

Some observations: (1) the solution to the non-homogeneous problem can be obtained from the solution to the homogeneous problem via Duhamel's principle; (2) if the term cu is not present, we know the exact solution to this IVP, (3) $u_t + cu = 0$ is equivalent to $e^{ct}u_t + ce^{ct}u = 0$ which converts to $(e^{ct}u)_t = 0$ and leads to $u = Ce^{-ct}$ as a solution (should remind you of an integrating factor technique).

From the above observations, we see that a good way to proceed is to multiply both sides by the function e^{ct}. This gives us:

$$e^{ct}u_t + ce^{ct}u - e^{ct}\Delta u = e^{ct}f$$

Since $\Delta(e^{ct}u) = e^{ct}\Delta u$, the above can be written as

$$(e^{ct}u)_t - \Delta(e^{ct}u) = e^{ct}f$$

and hence this converts into a regular heat equation formulation in terms of $e^{ct}u$, with non-homogeneous right hand side. The initial condition is unchanged, since $(e^{ct}u)(x, 0) = u(x, 0) = g(x)$. So now we can use Duhamel's principle and write the exact solution for this modified IVP as:

$$e^{ct}u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)e^{cs}f(y,s)dyds$$

The solution to the original IVP then follows:

$$u(x,t) = e^{-ct}\Big[\int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)e^{cs}f(y,s)dyds\Big]$$

#2

Now let us consider the usual heat equation $u_t = u_{xx}$ in 1d with $u(x, t)$ being a solution.

(a) Take $v(x, t) = u(x-y, t)$. How does one show this is also a solution? The easiest way is to plug it into the equation. First we need to use chain rule and compute partial derivatives: $v_t = (u(x - y, t))_t = u_t(x - y, t)$, $v_x = u_x(x - y, t)$, relying on the fact that $(x - y)_x = 1$. Since $u_t(x, t) = u_{xx}(x, t)$ for any $x \in \mathbb{R}$, we have $v_t = v_{xx}$. Notice that if we had an IVP originally with $u(x, 0) = g(x)$, the initial condition for v would be $h^y(x) = g(x - y)$, and the formula for the solution using fundamental function $\Phi(x, t)$ would give us:

$$u(x-y,t) = \int_{\mathbb{R}} \Phi(x-y-s,t)h^y(s)ds$$

where $s' = y + s$

(b) For any derivative D^α, where α is a multiindex, and the derivative is taken either in t or in x, we have $(D^\alpha u)_t = D^\alpha u_t = D^\alpha u_{xx} = (D^\alpha u)_{xx}$ since derivatives commute. it follows that any derivative of the solution of the heat equation is again a solution.

(c) $(au + bv)_t = au_t + bv_t = au_{xx} + bv_{xx} = (au + bv)_{xx}$, hence $au + bv$ is again a solution, if u, v are solutions.

(d) This follows from the fact that the operations of integration and differentiation are commutative:

$$\frac{\partial}{\partial t}\Big(\int_0^x u(y,t)dy \Big) = \int_0^x u_t(y,t)dy = \int_0^x u_{xx}(y,t)dy = \frac{\partial^2}{\partial x^2}\Big(\int_0^x u(y,t)dy \Big)$$

(e) Let $v(x,t) = u(\sqrt{a}x, at)$. Then $v_t(x,t) = av(x,t), v_x(x,t) = \sqrt{a}v(x,t), v_{xx} = av(x,t)$. It follows that $v_t = v_{xx}$.

#3

The easiest example of a Dirichlet problem with no solution can be constructed as follows. Let $U_T = U \times (0,T)$ and $\Sigma = \bar{U}_T - U_T$ be the boundary of this cylinder including the top, bottom and the sides, while denoting Γ_T to be the parabolic boundary comprised of the bottom and vertical sides only. Consider

$$\begin{aligned} u_t - \Delta u = 0, &\quad \text{in} \quad U_T \\ u = f, &\quad \text{on} \quad \Sigma \end{aligned}$$

Suppose $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves this IVP. Then it should satisfy the weak maximum principle, namely

$$\max_{\bar{U}_T} u(x,t) = \max_{\Gamma_T} u(x,t)$$

Since $u(x,t) = f(x)$ on Γ_T, we need the solution to satisfy $\max\limits_{\bar{U}_T} u = \max\limits_{\Gamma_T} f$. In particular, $f(x, t = T) < \max\limits_{\Gamma_T} f$, which does not hold for all continuous functions.

If we allow $t = T$ to be part of the cylinder $U_T = U \times (0,T]$, we need to construct a more elaborate example of a Dirichlet problem with no solution. One such example is a ball in $\mathbb{R}^3$ with deformable surface. We can push in a sharp spike at some point on this surface and assume that near the tip of the spike the surface takes the form of a conical surface obtained by rotating the curve

$$y = \begin{cases} e^{-1/x}. & \text{for } x > 0 \\ 0. & \text{for } x = 0 \end{cases}$$

about the x-axis. Then we can consider heat conduction on the interior of the deformed ball defined this way, called Ω. If the temperature distribution on $\partial\Omega$ is given by a continuous function f which is equal to zero at points of the spike and is equal to a large positive constant temperature T at points away from the spike, the seady state temperature $u(x)$ should be close to T for all x in Ω. But this is impossible, since $u(x)$ won't be able to approach the zero temperature as x approaches the spike from within of Ω. Basically, the spike doe not have enough surface area to keep the temperature at surrounding points close to zero, hence the solution fails to be continuous in the closure $\bar{\Omega}$. More details about the subject are available in Helms "Introduction to potential theory" (1975).

#4

Consider $v(x,t) = k(x,t)u(x/t, -1/t), t > 0$. To see that this solved the heat equation, let us compute its partial derivatives, using chain rule and product rule:

$$v_t(x,t) = k_t(x,t)u(\frac{x}{t}, -\frac{1}{t}) + k(x,t)(\frac{1}{t^2})u_t(\frac{x}{t}, -\frac{1}{t}) - k(x,t)(\frac{x}{t^2})u_x(\frac{x}{t}, -\frac{1}{t})$$

$$v_x(x,t) = k_x(x,t)u(\frac{x}{t}, -\frac{1}{t}) + k(x,t)(\frac{1}{t})u_x(\frac{x}{t}, -\frac{1}{t})$$

$$v_{xx}(x,t) = k_{xx}(x,t)u(\frac{x}{t}, -\frac{1}{t}) + 2k_x(x,t)(\frac{1}{t})u_x(\frac{x}{t}, -\frac{1}{t}) + k(x,t)(\frac{1}{t^2})u_{xx}(\frac{x}{t}, -\frac{1}{t})$$

Now using the fact that $k_t = k_{xx}, u_t = u_{xx}$, we can simplify this to

$$v_t - v_{xx} = -\frac{1}{t}\Big(\frac{xk(x,t)}{t} + 2k_x(x,t)\Big)\Big)u_x(\frac{x}{t}, -\frac{1}{t})$$

Using the definition of $k(x,t)$, we can easily see that $k_x(x,t) = -(x/2t)k(x,t)$, so that $xk(x,t) + 2tk_x(x,t) = 0$. This confirms the claim that $v(x,t)$ solves the heat equation. Since $u(x,t)$ was defined for all $t < 0$, $s = -1/t$ covers the domain $(0, \infty)$. In other words, $v(x,t)$ is a solution for all $t > 0$.