Math 678. Homework 2 Solutions.

#3, p.85
Following the proof of the mean value property, denote $\phi(r) = \fint_{\partial B(0,r)} u(y)dS(y)$ and use the same argument to show that $\phi'(r) = \fint_{B(0,r)} \Delta u(y)dS = -\fint_{B(0,r)} f(y)dS$.

Then fix some $\epsilon > 0$ and notice that $\phi(r) - \phi(\epsilon) = \int_\epsilon^r \phi'(s)ds$, with $\phi(\epsilon) \to u(0)$ as $\epsilon \to 0$. Since $u = g$ on $\partial B(0,r)$, $\phi(r) = \fint_{\partial B(0,r)} g(y)dS(y)$.

We can use polar coordinates and interchange the order of integration (leaving out the term that vanishes as $\epsilon \to 0$) to compute the righthand side:

$$-\int_\epsilon^r \phi'(s)ds = \int_\epsilon^r \frac{1}{s^{n-1}n\alpha(n)} \int_{B(0,s)} f(y)dSds =$$
$$\frac{1}{n\alpha(n)} \int_\epsilon^r \int_0^s \int_{\partial B(0,t)} \frac{f(y)}{s^{n-1}} dy\,dt\,ds = \frac{1}{n\alpha(n)} \int_\epsilon^r \int_t^r \int_{\partial B(0,t)} \frac{f(y)}{s^{n-1}} dy\,ds\,dt =$$
$$\frac{1}{n\alpha(n)} \int_\epsilon^r \int_{\partial B(0,t)} \int_t^r \frac{f(y)}{s^{n-1}} ds\,dy\,dt = \frac{1}{n\alpha(n)} \int_\epsilon^r \int_{\partial B(0,t)} f(y) \int_t^r \frac{1}{s^{n-1}} ds\,dy\,dt =$$
$$-\frac{1}{n(n-2)\alpha(n)} \int_\epsilon^r \int_{\partial B(0,t)} f(y)\Big[\frac{1}{r^{n-2}} - \frac{1}{t^{n-2}}\Big] dy\,dt =$$
$$\frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} f(y)\Big[\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}}\Big] dy$$

As $\epsilon \to 0$, this leaves us with

$$u(0) = \fint_{\partial B(0,r)} g(y)dS(y) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} f(y)\Big[\frac{1}{r^{n-2}} - \frac{1}{|y|^{n-2}}\Big] dy$$

#5, p.85 (a) Using the mean value property argument, we can show that for the same $\phi(r)$ as defined above, $\phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta v(y)dy \geq 0$. Let us write $\phi(r) - \phi(0) = \int_0^r \phi'(s)ds \geq 0$, which means that $\fint_{\partial B(x,r)} v(y)dS(y) = \phi(r) \geq \phi(0) = v(x)$. Then notice that

$$\int_{B(x,r)} v(y)dS(y) = \int_0^r \int_{\partial B(x,s)} v(y)dS(y)ds =$$
$$\int_0^r n\alpha(n)s^{n-1}\Big[\fint_{\partial B(x,s)} v(y)dS(y)\Big] ds \geq$$
$$n\alpha(n)\Big[\int_0^r s^{n-1}ds\Big] v(x) = n\alpha(n)r^n v(x)/n = \alpha(n)r^n v(x)$$

The conclusion follows.

(b) Suppose there is an interior maximum for $v(x)$ at x_0. Then $u(x_0) = \max_{\bar{U}} u = M$ and by (a) for any ball $B(x_0,r)$, $M = u(x_0) \leq \fint_{B(x_0,r)} v(y)dy \leq$

M. Equality is achieved when $u \equiv M$ on $B(x_0, r)$. For a connected domain, it follows that $u \equiv M$ on $\bar{U}$, so maximum principle holds.

(c) Let $v(x) = f(u(x))$ with f-convex. Any convex function f is continuous, and satisfies Jensen's inequality on a bounded domain G:

$$f(\fint_G u(x)dx) \le \fint_G f(u(x))dx$$

If u is harmonic, $u(x) = \fint_{B(x,r)} u(y)dy$, so from Jensen's inequality, $f(u(x)) \le \fint_{B(x,r)} f(u(y))dy$. So (a) holds even if the function is not smooth.

To show that the function is subharmonic according to the definition $-\Delta(f(u(x))) \le 0$ in the case the function $f \in C^2(U)$, we can use direct computation:

$$-\Delta v = -\sum_{i=1}^{n} v_{x_i x_i} = -\sum_{i=1}^{n} f''(u)u_{x_i}^2 - \sum_{i=1}^{n} f'(u)u_{x_i x_i} = -f''(u)\sum_{i=1}^{n} u_{x_i}^2 - f'(u)\Delta u \le 0.$$

(d) You can notice $|Du|$ is harmonic for a harmonic function u (direct but tedious computation). Since $f(x) = |x|^2$ is a convex function on $\mathbb{R}$, we can use the result of (c), which proves that the function $|Du|^2$ is subharmonic.

You can also show this by direct computation:

$$v_{x_i x_i} = \sum_{k=1}^{n} (2u_{x_i x_i}^2 + 2u_{x_k x_i x_i}) \ge 2\sum_{k=1}^{n} u_{x_k x_i x_i}.$$

Summing over all i and noticing that $\sum_i \sum_k u_{x_k x_i x_i} = \sum_k \frac{\partial}{\partial x_k} \sum_i u_{x_i x_i} = 0$, we get the conclusion.

Part II.

Consider any ball $B(x, r) \subset U$, a function $v \in C(U)$ such that $v(x) \le \fint_{B(x,r)} v dy$ and a harmonic function u defined on a domain that includes this ball. Consider the difference $w = v - u$. Clearly, $w \le 0$ on $\partial B(x, r)$. We need to show $w \le 0$ in $B(x, r)$.

The function w is subharmonic on $B(x, r)$ in the sense that $w(x) = u(x) - v(x) \le \fint_{B(x,r)} w(y)dy$, continuous on $B(x, r)$ and hence satisfies the strong, and hence the weak maximum principle, as shown in 5(b) above. So $\max_{\bar{B}(x,r)}(w) = \max_{\partial B(x,r)} w = 0$, as needed.