

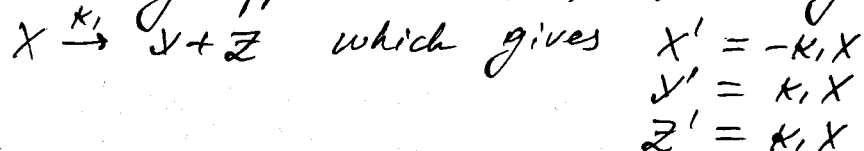
Math 413.

Homework 4 Solutions.

#3.2
$$\begin{cases} X' = aX + bYZ \\ Y' = cX + bYZ \\ Z' = cX + dYZ \end{cases}$$

The aX, cX terms require reaction of the form $X \rightarrow \text{product}$

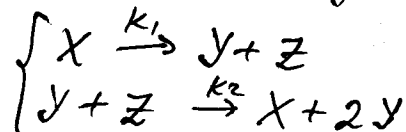
Since they appear in X', Y', Z' , we may consider



The bYZ, dYZ terms come from Y and Z being reactants (left-hand side terms), so we need

$Y + Z \rightarrow \text{products}$. Since X' also involves bYZ , we consider $Y + Z \xrightarrow{k_2} X$. This produces $\begin{aligned} X' &= +k_2 YZ \\ Y' &= -k_2 YZ \\ Z' &= -k_2 YZ \end{aligned}$

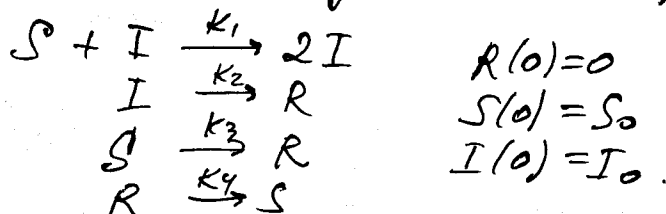
Since original system has bYZ in both X' and Y' , we need an additional product term $2Y$ to balance $-k_2 YZ$, so the final system becomes:



From this argument we see that $\begin{aligned} a &= -k_1 \\ c &= k_1 \Rightarrow a = -c \\ b &= k_2 \\ d &= -k_2 \Rightarrow b = -d \end{aligned}$

Same conclusion can be reached from a more general argument starting with $\alpha X + \beta Y + \gamma Z \rightarrow \delta_1 X + \beta_1 Y + \delta_2 Z$.

#3.8



- (a) We assume all susceptible will become ill on contact, all ill eventually recover, but can get sick again. Susceptible can recover after vaccination.

$$(b) \begin{cases} \frac{dS}{dt} = -K_1 SI - K_3 S + K_4 R \\ \frac{dI}{dt} = K_1 SI - K_2 I \\ \frac{dR}{dt} = K_2 I + K_3 S - K_4 R \end{cases}$$

Conservation law: $S + I + R = S_0 + I_0 \Rightarrow R = S_0 + I_0 - S - I$

Reduced system:

$$\begin{cases} \frac{dS}{dt} = -K_1 SI - K_3 S + K_4 (S_0 + I_0 - S - I) \\ \frac{dI}{dt} = K_1 SI - K_2 I \end{cases}$$

(c) Scaling: $\begin{matrix} S = N_0 \cdot s \\ I = N_0 \cdot i \\ t = t_c \cdot \tau \end{matrix}$ leads to

$$\begin{cases} \frac{N_0}{t_c} \frac{ds}{d\tau} = -K_1 N_0^2 si - K_3 N_0 s + K_4 \underbrace{(S_0 + I_0)}_{N_0} - N_0 s - N_0 i \\ \frac{N_0}{t_c} \frac{di}{d\tau} = K_1 N_0^2 si - K_2 N_0 i \end{cases}$$

$$\Rightarrow \begin{cases} \frac{ds}{d\tau} = -K_1 t_c N_0 si - K_3 t_c s + K_4 t_c (1 - s - i) \\ \frac{di}{d\tau} = K_1 t_c N_0 si - K_2 t_c i \end{cases}$$

$$s(0) = \frac{S_0}{N_0}, \quad i(0) = \frac{I_0}{N_0}$$

$$\pi_1 = K_1 t_c N_0, \quad \pi_2 = K_2 t_c, \quad \pi_3 = K_3 t_c, \quad \pi_4 = K_4 t_c.$$

$$\Rightarrow \text{Put } \pi_1 = K_1 t_c N_0 = 1 \Rightarrow t_c = \frac{1}{K_1 N_0},$$

$$\pi_2 = \frac{K_2}{K_1 N_0} = \alpha$$

$$\pi_3 = \frac{K_3}{K_1 N_0} = \beta, \quad \pi_4 = \frac{K_4}{K_1 N_0} = \delta$$

So the system becomes:
$$\begin{cases} \frac{ds}{d\tau} = -si - \beta s + \delta(1 - s - i) \\ \frac{di}{d\tau} = si - \alpha i \end{cases}$$

(d) Steady states satisfy:

$$\begin{cases} -si - \beta s + \delta(1 - s - i) = 0 \\ (s - \alpha)i = 0 \end{cases}$$

Case 1. $i=0$ from 2nd equation

$$\Rightarrow -\beta s + \gamma(1-s) = 0$$

$$\gamma = (\beta + \gamma)s \Rightarrow s = \frac{\gamma}{\beta + \gamma} \Rightarrow \left(\frac{\gamma}{\beta + \gamma}, 0 \right)$$

first steady state

Case 2: $i \neq 0, s = d$

$$\Rightarrow -di - \beta d + \gamma(1-d-i) = 0$$

$$i(-d + \gamma) - \beta d + \gamma(1-d) = 0$$

$$i = \frac{\gamma(1-d) - \beta d}{d + \gamma}$$

2nd steady state: $\left(\underset{s_s}{d}, \underset{i_s}{\frac{\gamma - \gamma d - \beta d}{d + \gamma}} \right)$ [epidemic equilibrium]

(e) Since $S + I + R = N_0$ from conservation law

$$\Rightarrow \underset{\frac{S}{N_0}}{s} + \underset{\frac{I}{N_0}}{i} + \frac{R}{N_0} = 1 \text{ and } s, i, R \geq 0, \text{ so } 0 \leq s \leq 1$$

$$0 \leq i \leq 1.$$

This means that $0 \leq \frac{\gamma - \gamma d - \beta d}{d + \gamma} \leq 1$ for the epidemic equilibrium to be feasible.

(f) Asymptotic stability of $\left(\frac{\gamma}{\beta + \gamma}, 0 \right)$:

$$J = \text{Jacobian} = \begin{bmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial i} \\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial i} \end{bmatrix} = \begin{bmatrix} -i - \beta - \gamma & -s - \gamma \\ i & s - d \end{bmatrix}$$

$$f_1 = -s_i - \beta s + \gamma(1-s-i)$$

$$f_2 = si - di$$

$$\text{If } i_s = 0, s_s = \frac{\gamma}{\beta + \gamma} \Rightarrow J = \begin{bmatrix} -\beta - \gamma & -\frac{\gamma}{\beta + \gamma} - \gamma \\ 0 & \frac{\gamma}{\beta + \gamma} - d \end{bmatrix}$$

Asymp. stable if eigenvalues have $\text{Re}(r_i) < 0$.

Here $r_1 = -\beta - \gamma < 0$ always

$$r_2 = \frac{\gamma}{\beta + \gamma} - d < 0 \Rightarrow \boxed{d > \frac{\gamma}{\beta + \gamma}} \text{ assumption necessary for stability of } i_s = 0.$$

(g) Epidemic equilibrium $\left(\underset{s_s}{\alpha}, \underset{i_s}{\frac{\sigma - \sigma \alpha - \beta \alpha}{\sigma + \alpha}} \right)$

is asymp. stable if eigenvalues of J have $\text{Re}(r_i) < 0$.

$$J = \begin{bmatrix} -i_s - \beta - \sigma & -\alpha - \sigma \\ i_s & 0 = \alpha - \alpha \end{bmatrix}$$

$$|J - rI| = r(i_s + \beta + \sigma + r) + i_s(\alpha + \sigma) = 0$$

$$r^2 + (i_s + \beta + \sigma)r + i_s(\alpha + \sigma) = 0$$

$$r = \frac{-(i_s + \beta + \sigma) \pm \sqrt{(i_s + \beta + \sigma)^2 - 4i_s(\alpha + \sigma)}}{2}$$

Case 1: real roots $\Rightarrow$ both negative ~~sing~~

Since $\frac{-(i_s + \beta + \sigma) + \sqrt{(i_s + \beta + \sigma)^2 - 4i_s(\alpha + \sigma)}}{2} < 0$
 $(i_s > 0, \alpha + \sigma > 0)$

Case 2: complex roots $\Rightarrow \text{Re}(r) = \frac{-(i_s + \beta + \sigma)}{2} < 0$

Since $i_s, \beta, \sigma > 0$.

(assuming (s_s, i_s) is feasible
 i.e. condition in (e) holds.)

In other words, epidemic equilibrium is always asymp. stable assuming $i_s > 0$.

(h) To avoid epidemic, we need to ensure infeasibility of the second equilibrium and asymp. stability of the other one ($i_s = 0$).

Infeasibility : $\boxed{\sigma - \sigma \alpha - \beta \alpha < 0}$ (from (e))

asymp. stability of $i_s = 0$: $\boxed{\alpha > \frac{\sigma}{\beta + \sigma}}$ (from (f))

these two are actually equivalent!

So we need to enforce $\sigma - \sigma \alpha - \beta \alpha < 0$

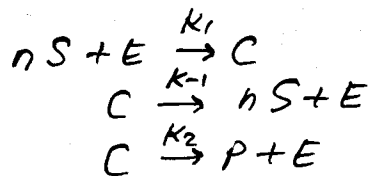
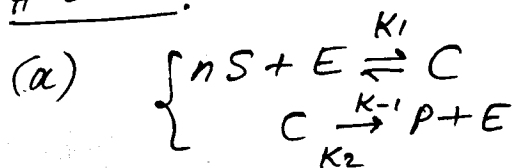
$$\frac{K_4}{K_1 N_0} - \frac{K_4 K_2}{(K_1 N_0)^2} - \frac{K_3 K_2}{(K_1 N_0)^2} < 0$$

$$\Rightarrow K_1 K_4 N_0 - K_2 K_4 - K_3 K_2 < 0$$

$$\boxed{K_3 > \frac{K_1 K_4 N_0 - K_2 K_4}{K_2}}$$

Vaccination rate constant
 should satisfy this inequality.

3.15.



$$\begin{cases} \frac{dS}{dt} = -nK_1 S^n E + nK_{-1} C \\ \frac{dE}{dt} = -K_1 S^n E + K_{-1} C + K_2 C \\ \frac{dC}{dt} = K_1 S^n E - K_{-1} C - K_2 C \\ \frac{dP}{dt} = K_2 C \end{cases}$$

(b) Cons. law: $E + C = E_0 \Rightarrow E = E_0 - C$

$$\Rightarrow \begin{cases} \frac{dS}{dt} = -nK_1 S^n (E_0 - C) + nK_{-1} C \\ \frac{dC}{dt} = K_1 S^n (E_0 - C) - K_{-1} C - K_2 C \end{cases}$$

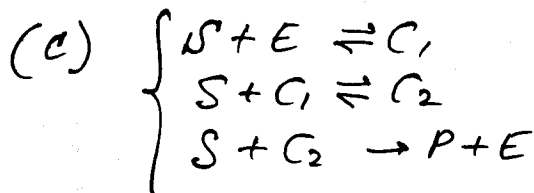
QSSA: $\frac{dC}{dt} = 0 = K_1 S^n (E_0 - C) - K_{-1} C - K_2 C \Rightarrow$

$$(K_1 S^n + K_{-1} + K_2) C = K_1 E_0 S^n$$

$$C = \frac{K_1 E_0 S^n}{K_{-1} + K_2 + K_1 S^n}$$

Hence $\frac{dP}{dt} = \frac{K_1 K_2 E_0 S^n}{K_{-1} + K_2 + K_1 S^n} = \frac{a S^n}{b + S^n}$

To get $\frac{dP}{dt} = \frac{a S^n}{b + S^n}$, take $a = K_2 E_0$, $b = \frac{K_{-1} + K_2}{K_1}$
(Hill equation).



$$\begin{cases} \frac{dS}{dt} = -K_1 S E + K_{-1} C_1 - K_2 S C_1 + K_{-2} C_2 - K_3 S C_2 \\ \frac{dE}{dt} = -K_1 S E + K_{-1} C_1 + K_3 S C_2 \\ \frac{dC_1}{dt} = K_1 S E - K_{-1} C_1 - K_2 S C_1 + K_{-2} C_2 \\ \frac{dC_2}{dt} = K_2 S C_1 - K_{-2} C_2 - K_3 S C_2 \\ \frac{dP}{dt} = K_3 S C_2 \end{cases}$$

Cons. law: $E + C_1 + C_2 = E_0$

$\Rightarrow E = E_0 - C_1 - C_2$

QSSA: $\begin{cases} \frac{dC_1}{dt} = 0 = K_1 S(E_0 - C_1 - C_2) - K_{-1} C_1 - K_2 S C_1 + K_{-2} C_2 \Rightarrow \\ \frac{dC_2}{dt} = 0 = K_2 S C_1 - K_{-2} C_2 - K_3 S C_2 \end{cases}$

Rewrite: $\begin{cases} (-K_1 S - K_{-1} - K_2 S) C_1 + (K_{-2} - K_1 S) C_2 = -K_1 E_0 S \\ K_2 S C_1 - (K_{-2} + K_3 S) C_2 = 0 \end{cases}$

$C_1 = \frac{(K_{-2} + K_3 S) C_2}{K_2 S}$

$\frac{-(K_1 S + K_{-1} + K_2 S)(K_{-2} + K_3 S) + K_2 S(K_{-2} - K_1 S)}{K_2 S} \cdot C_2 = -K_1 E_0 S$

$C_2 = \frac{-K_1 K_2 E_0 S^2}{K_2 S(K_{-2} - K_1 S) - (K_1 S + K_{-1} + K_2 S)(K_{-2} + K_3 S)}$
 $= \frac{-K_1 K_2 E_0 S^2}{-K_1 K_2 S^2 + K_2 K_{-2} S - K_1 K_{-2} S - K_1 K_3 S^2 - K_{-1} K_{-2} - K_{-1} K_3 S - K_2 K_{-2} S - K_2 K_3 S^2}$
 $= \frac{K_1 K_2 E_0 S^2}{(K_1 K_2 + K_1 K_3 + K_2 K_3) S^2 + (K_1 K_{-2} + K_{-1} K_3) S + K_{-1} K_{-2}}$

Hence $\frac{dP}{dt} = \frac{K_1 K_2 K_3 E_0 S^3}{(K_1 K_2 + K_1 K_3 + K_2 K_3) S^2 + (K_1 K_{-2} + K_{-1} K_3) S + K_{-1} K_{-2}}$

which has the form $\frac{aS^3}{b+S^2}$ under assumption $(K_1 K_{-2} + K_{-1} K_3) S \ll \alpha S^2$

with $a = \frac{K_1 K_2 K_3 E_0}{\alpha}$ where $\rightarrow \alpha = K_1 K_2 + K_1 K_3 + K_2 K_3$
 $b = \frac{K_{-1} K_{-2}}{\alpha}$

Part II.

(a) $X + Y \xrightarrow{k} Z$

$$\frac{dX}{dt} = -kXY$$

$$\frac{dY}{dt} = -kXY$$

$$\Rightarrow X - Y = X(0) - Y(0) = X_0 - Y_0$$

$$Y = X - X_0 + Y_0$$

$$\Rightarrow \frac{dX}{dt} = -kX(X - (X_0 - Y_0)) = kX(X_0 - Y_0 - X)$$

$$\frac{dX}{dt} = (X_0 - Y_0)kX \left(1 - \frac{X}{X_0 - Y_0}\right) \quad \text{assuming } X_0 - Y_0 \neq 0.$$

$$\Rightarrow \frac{dX}{dt} = rX \left(1 - \frac{X}{K}\right), \quad \text{where } r = (X_0 - Y_0)k$$

logistic equation $K = X_0 - Y_0$

(b) $A \xrightleftharpoons[k_{-1}]{k_1} B \quad A(0) = A_0, B(0) = B_0.$

$$\frac{dA}{dt} = -k_1A + k_{-1}B$$

$$\frac{dB}{dt} = k_1A - k_{-1}B$$

Cons. law: $A + B = A_0 + B_0 \Rightarrow B = A_0 + B_0 - A$

$$\Rightarrow \frac{dA}{dt} = -k_1A + k_{-1}(A_0 + B_0 - A) = -(k_1 + k_{-1})A + k_{-1}(A_0 + B_0)$$

This equation is of the form $A' + \alpha A = \beta$

$$\Rightarrow \frac{dA}{dt} = -\alpha \left(A - \frac{\beta}{\alpha}\right)$$

$\alpha = k_1 + k_{-1}, \beta = k_{-1}(A_0 + B_0)$
Can be solved by separation of variables for instance (or by integrating factor).

$$\int \frac{dA}{A - \frac{\beta}{\alpha}} = -\int \alpha dt \Rightarrow \ln \left| A - \frac{\beta}{\alpha} \right| = -\alpha t + C$$

$$\left| A - \frac{\beta}{\alpha} \right| = Ce^{-\alpha t} \Rightarrow A = \frac{\beta}{\alpha} + Ce^{-\alpha t}$$

Hence $A(t) = \frac{\beta}{\alpha} + (A_0 - \frac{\beta}{\alpha})e^{-\alpha t}$

$$A(0) = \frac{\beta}{\alpha} + C = A_0$$

$$\Rightarrow C = A_0 - \frac{\beta}{\alpha}$$

Where $\alpha = k_1 + k_{-1}$

$$\beta = k_{-1}(A_0 + B_0).$$