Math 413.

Homework 4 Solutions.

#3.2
$$\begin{cases} X' = aX + bYZ \\ Y' = cX + bYZ \\ Z' = cX + dYZ \end{cases}$$

The aX, cX terms require reaction of the form $X \to$ product. Since they appear in X', Y', Z', we may consider $X \xrightarrow{k_1} Y + Z$ which gives
$$X' = -k_1 X$$
$$Y' = k_1 X$$
$$Z' = k_1 X$$

The bYZ, dYZ terms come from Y and Z being reactants (left-hand side terms), so we need $Y + Z \to$ products. Since X' also involves bYZ, we consider $Y + Z \xrightarrow{k_2} X$. This produces
$$X' = +k_2 YZ$$
$$Y' = -k_2 YZ$$
$$Z' = -k_2 YZ$$

Since original system has bYZ in both X' and Y', we need an additional product term $2Y$ to balance $-k_2 YZ$, so the final system becomes:
$$\begin{cases} X \xrightarrow{k_1} Y + Z \\ Y + Z \xrightarrow{k_2} X + 2Y \end{cases}$$

From this argument we see that
$$a = -k_1, \quad c = k_1 \Rightarrow a = -c$$
$$b = k_2, \quad d = -k_2 \Rightarrow b = -d$$

Same conclusion can be reached from a more general argument starting with $\alpha X + \beta Y + \gamma Z \to \alpha_1 X + \beta_1 Y + \gamma_1 Z$.

#3.8
$$S + I \xrightarrow{k_1} 2I$$
$$I \xrightarrow{k_2} R$$
$$S \xrightarrow{k_3} R$$
$$R \xrightarrow{k_4} S$$

$R(0) = 0$
$S(0) = S_0$
$I(0) = I_0$.

(a) We assume all susceptible will become ill on contact, all ill eventually recover, but can get sick again. Susceptible can recover after vaccination.

(b) $$\begin{cases} \frac{dS}{dt} = -k_1 SI - k_3 S + k_4 R \\ \frac{dI}{dt} = k_1 SI - k_2 I \\ \frac{dR}{dt} = k_2 I + k_3 S - k_4 R \end{cases}$$

Conservation law: $S+I+R = S_0+I_0 \Rightarrow R = S_0+I_0-S-I$

Reduced system:

$$\begin{cases} \frac{dS}{dt} = -k_1 SI - k_3 S + k_4 (S_0+I_0-S-I) \\ \frac{dI}{dt} = k_1 SI - k_2 I \end{cases}$$

(c) Scaling: $S = N_0 \cdot s$, $I = N_0 \cdot i$, $t = t_c \cdot \tau$ leads to

$$\begin{cases} \frac{N_0}{t_c}\frac{ds}{d\tau} = -k_1 N_0^2 si - k_3 N_0 s + k_4 (\underbrace{S_0+I_0}_{N_0} - N_0 s - N_0 i) \\ \frac{N_0}{t_c}\frac{di}{d\tau} = k_1 N_0^2 si - k_2 N_0 i \end{cases}$$

$$\Rightarrow \begin{cases} \frac{ds}{d\tau} = -k_1 t_c N_0 si - k_3 t_c s + k_4 t_c (1-s-i) \\ \frac{di}{d\tau} = k_1 t_c N_0 si - k_2 t_c i \end{cases}$$

$$s(0) = \frac{S_0}{N_0}, \quad i(0) = \frac{I_0}{N_0}$$

$\Pi_1 = k_1 t_c N_0$, $\Pi_2 = k_2 t_c$, $\Pi_3 = k_3 t_c$, $\Pi_4 = k_4 t_c$.

$\Rightarrow$ Put $\Pi_1 = k_1 t_c N_0 = 1 \Rightarrow t_c = \frac{1}{k_1 N_0}$,

$\Pi_2 = \frac{k_2}{k_1 N_0} = \alpha$

$\Pi_3 = \frac{k_3}{k_1 N_0} = \beta$, $\Pi_4 = \frac{k_4}{k_1 N_0} = \gamma$

So the system becomes:

$$\begin{cases} \frac{ds}{d\tau} = -si - \beta s + \gamma(1-s-i) \\ \frac{di}{d\tau} = si - \alpha i \end{cases}$$

(d) Steady states satisfy:

$$\begin{cases} -si - \beta s + \gamma(1-s-i) = 0 \\ (s-\alpha) i = 0 \end{cases}$$

Case 1. $i=0$ from 2nd equation

$$\Rightarrow \quad -\beta s+\delta(1-s)=0$$
$$\delta=(\beta+\delta)s \Rightarrow s=\frac{\delta}{\beta+\delta} \Rightarrow \Big(\underset{s_s}{\frac{\delta}{\beta+\delta}},\underset{i_s}{0}\Big)$$

first steady state

Case 2: $i\neq 0$, $s=\alpha$

$$\Rightarrow \quad -\alpha i-\beta\alpha+\delta(1-\alpha-i)=0$$
$$i(-\alpha-\delta)-\beta\alpha+\delta(1-\alpha)=0$$
$$i=\frac{\delta(1-\alpha)-\beta\alpha}{\alpha+\delta}$$

2nd steady state: $\Big(\underset{s_s}{\alpha},\ \underset{i_s}{\frac{\delta-\delta\alpha-\beta\alpha}{\delta+\alpha}}\Big)$ [epidemic equilibrium]

(e) Since $S+I+R=N_0$ from conservation law

$\Rightarrow \quad \underset{S/N_0}{s}+\underset{I/N_0}{i}+\frac{R}{N_0}=1$ and $s,i,R\geq 0$, so $0\leq s\leq 1$, $0\leq i\leq 1$.

This means that $\underbrace{0\leq\frac{\delta-\delta\alpha-\beta\alpha}{\delta+\alpha}\leq 1}$ for the epidemic equilibrium to be feasible.

(f) Asymptotic stability of $\left(\frac{\delta}{\beta+\delta},0\right)$:

$$J=\text{Jacobian}=\begin{bmatrix}\frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial i}\\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial i}\end{bmatrix}=\begin{bmatrix}-i-\beta-\delta & -s-\delta\\ i & s-\alpha\end{bmatrix}$$

$f_1=-si-\beta s+\delta(1-s-i)$

$f_2=si-\alpha i$

If $i_s=0$, $S_s=\frac{\delta}{\beta+\delta}$ $\Rightarrow J=\begin{bmatrix}-\beta-\delta & -\frac{\delta}{\beta+\delta}-\delta\\ 0 & \frac{\delta}{\beta+\delta}-\alpha\end{bmatrix}$

Asymp. stable if eigenvalues (r_i) have $Re(r_i)<0$.

Here $r_1=-\beta-\delta<0$ always

$$r_2=\frac{\delta}{\beta+\delta}-\alpha<0 \Rightarrow \boxed{\alpha>\frac{\delta}{\beta+\delta}}$$

assumption necessary for stability of $i_s=0$.

(g) Epidemic equilibrium $\left(\underset{S_s}{\overset{\alpha}{\|}}, \underset{i_s}{\overset{\frac{\delta - \delta\alpha - \beta\alpha}{\delta + \alpha}}{\|}}\right)$

is asymp. stable if eigenvalues of J have $Re(r_i) < 0$.

$$J = \begin{bmatrix} -i_s - \beta - \delta & -\alpha - \delta \\ i_s & 0 = \alpha - \alpha \end{bmatrix}$$

$$|J - rI| = r(i_s + \beta + \delta + r) + i_s(\alpha + \delta) = 0$$

$$r^2 + (i_s + \beta + \delta) r + i_s(\alpha + \delta) = 0$$

$$r = \frac{-(i_s + \beta + \delta) \pm \sqrt{(i_s + \beta + \delta)^2 - 4 i_s(\alpha + \delta)}}{2}$$

Case 1: real roots $\Rightarrow$ both negative ~~sing~~
since $\underset{0}{-(i_s + \beta + \delta) + \sqrt{(i_s + \beta + \delta)^2 - 4 i_s(\alpha + \delta)}}$
$(i_s > 0, \alpha + \delta > 0)$

Case 2: complex roots $\Rightarrow$ $Re(r) = -\frac{(i_s + \beta + \delta)}{2} < 0$

Since $i_s, \beta, \delta > 0$.
(assuming (S_s, i_s) is feasible i.e. condition in (e) holds.)

In other words, epidemic equilibrium is always asymp. stable assuming $i_s > 0$.

(h) To avoid epidemic, we need to ensure infeasibility of the second equilibrium and asymp. stability of the other one ($i_s = 0$).

Infeasibility : $\boxed{\delta - \delta\alpha - \beta\alpha < 0}$ (from (e))
asymp. stability of $i_s = 0$: $\boxed{\alpha > \frac{\delta}{\beta + \delta}}$ (from (f))

these two are actually equivalent!

So we need to enforce $\delta - \delta\alpha - \beta\alpha < 0$

$$\frac{K_4}{K_1 N_0} - \frac{K_4 K_2}{(K_1 N_0)^2} - \frac{K_3 K_2}{(K_1 N_0)^2} < 0$$

$$\Rightarrow K_1 K_4 N_0 - K_2 K_4 - K_3 K_2 < 0$$

$$\boxed{K_3 > \frac{K_1 K_4 N_0 - K_2 K_4}{K_2}}$$

Vaccination rate constant should satisfy this inequality.

3.15.

(a) $\begin{cases} nS + E \underset{}{\overset{K_1}{\rightleftharpoons}} C \\ C \overset{K_{-1}}{\underset{K_2}{\longrightarrow}} P + E \end{cases}$

$$nS + E \xrightarrow{K_1} C$$
$$C \xrightarrow{K_{-1}} nS + E$$
$$C \xrightarrow{K_2} P + E$$

$$\begin{cases} \frac{dS}{dt} = -nK_1 S^n E + nK_{-1} C \\ \frac{dE}{dt} = -K_1 S^n E + K_{-1} C + K_2 C \\ \frac{dC}{dt} = K_1 S^n E - K_{-1} C - K_2 C \\ \frac{dP}{dt} = K_2 C \end{cases}$$

(b) Cons. law: $E + C = E_0 \Rightarrow E = E_0 - C$

$$\Rightarrow \begin{cases} \frac{dS}{dt} = -nK_1 S^n (E_0 - C) + nK_{-1} C \\ \frac{dC}{dt} = K_1 S^n (E_0 - C) - K_{-1} C - K_2 C \end{cases}$$

QSSA: $\frac{dC}{dt} = 0 = K_1 S^n (E_0 - C) - K_{-1} C - K_2 C \Rightarrow$

$$(K_1 S^n + K_{-1} + K_2) C = K_1 E_0 S^n$$
$$C = \frac{K_1 E_0 S^n}{K_{-1} + K_2 + K_1 S^n}$$

Hence $\frac{dP}{dt} = \frac{K_1 K_2 E_0 S^n}{K_{-1} + K_2 + K_1 S^n} = \frac{\alpha S^n}{\beta + \alpha S^n}$

To get $\frac{dP}{dt} = \frac{a S^n}{b + S^n}$, take $a = K_2 E_0$, $b = \frac{K_{-1} + K_2}{K_1}$
(Hill equation).

(c) $\begin{cases} S + E \rightleftharpoons C_1 \\ S + C_1 \rightleftharpoons C_2 \\ S + C_2 \rightarrow P + E \end{cases}$

$$\begin{cases} \frac{dS}{dt} = -K_1 SE + K_{-1} C_1 - K_2 SC_1 + K_{-2} C_2 - K_3 SC_2 \\ \frac{dE}{dt} = -K_1 SE + K_{-1} C_1 + K_3 SC_2 \\ \frac{dC_1}{dt} = K_1 SE - K_{-1} C_1 - K_2 SC_1 + K_{-2} C_2 \\ \frac{dC_2}{dt} = K_2 SC_1 - K_{-2} C_2 - K_3 SC_2 \\ \frac{dP}{dt} = K_3 SC_2 \end{cases}$$

Cons. law: $\quad E + C_1 + C_2 = E_0$

$$\Rightarrow \quad E = E_0 - C_1 - C_2$$

QSSA: $\begin{cases} \dfrac{dC_1}{dt} = 0 = K_1 S(E_0 - C_1 - C_2) - K_{-1} C_1 - K_2 S C_1 + K_{-2} C_2 \Rightarrow \\[2mm] \dfrac{dC_2}{dt} = 0 = K_2 S C_1 - K_{-2} C_2 - K_3 S C_2 \end{cases}$

Rewrite: $\begin{cases} (-K_1 S - K_{-1} - K_2 S) C_1 + (K_{-2} - K_1 S) C_2 = -K_1 E_0 S \\ K_2 S \cdot C_1 - (K_{-2} + K_3 S) C_2 = 0 \end{cases}$

$$C_1 = \frac{(K_{-2} + K_3 S) C_2}{K_2 S}$$

$$\frac{-(K_1 S + K_{-1} + K_2 S)(K_{-2} + K_3 S) + K_2 S (K_{-2} - K_1 S)}{K_2 S} \cdot C_2 = -K_1 E_0 S$$

$$C_2 = \frac{-K_1 K_2 E_0 S^2}{K_2 S (K_{-2} - K_1 S) - (K_1 S + K_{-1} + K_2 S)(K_{-2} + K_3 S)}$$

$$= \frac{-K_1 K_2 E_0 S^2}{-K_1 K_2 S^2 + \cancel{K_2 K_{-2} S} - K_1 K_{-2} S - K_1 K_3 S^2 - K_{-1} K_{-2} - K_{-1} K_3 S - \cancel{K_2 K_{-2} S} - K_2 K_3 S^2}$$

$$= \frac{K_1 K_2 E_0 S^2}{(K_1 K_2 + K_1 K_3 + K_2 K_3) S^2 + (K_1 K_{-2} + K_{-1} K_3) S + K_{-1} K_{-2}}$$

Hence $$\frac{dP}{dt} = \frac{K_1 K_2 K_3 E_0 S^3}{(K_1 K_2 + K_1 K_3 + K_2 K_3) S^2 + (K_1 K_{-2} + K_{-1} K_3) S + K_{-1} K_{-2}}$$

which has the form $\dfrac{a S^3}{b + S^2}$ under assumption

$$(K_1 K_{-2} + K_{-1} K_3) S \ll \alpha S^2$$

where $\rightarrow$ $\alpha = K_1 K_2 + K_1 K_3 + K_2 K_3$

with $a = \dfrac{K_1 K_2 K_3 E_0}{\alpha}$

$b = \dfrac{K_{-1} K_{-2}}{\alpha}$

Part II.

(a) $X + Y \xrightarrow{k} Z$

$$\frac{dX}{dt} = -kXY$$
$$\frac{dY}{dt} = -kXY$$
$$\Rightarrow \quad X - Y = X(0) - Y(0) = X_0 - Y_0$$
$$Y = X - X_0 + Y_0$$

$$\Rightarrow \frac{dX}{dt} = -kX(X - (X_0 - Y_0)) = kX(X_0 - Y_0 - X)$$

$$\frac{dX}{dt} = (X_0 - Y_0)kX\left(1 - \frac{X}{X_0 - Y_0}\right)$$ assuming $X_0 - Y_0 \neq 0$.

$$\Rightarrow \underbrace{\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right)}_{\text{logistic equation}}, \quad \text{where } r = (X_0 - Y_0)k, \quad K = X_0 - Y_0$$

(b) $A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B \qquad A(0) = A_0, \; B(0) = B_0.$

$$\begin{cases} \frac{dA}{dt} = -k_1 A + k_{-1} B \\ \frac{dB}{dt} = k_1 A - k_{-1} B \end{cases}$$

Cons. law: $A + B = A_0 + B_0 \Rightarrow B = A_0 + B_0 - A$

$$\Rightarrow \frac{dA}{dt} = -k_1 A + k_{-1}(A_0 + B_0 - A) = -(k_1 + k_{-1})A + k_{-1}(A_0 + B_0)$$

This equation is of the form $A' + \alpha A = \beta$

$\alpha = k_1 + k_{-1}, \; \beta = k_{-1}(A_0 + B_0)$

can be solved by separation of variables for instance (or by integrating factor).

$$\Rightarrow \frac{dA}{dt} = -\alpha\left(A - \frac{\beta}{\alpha}\right)$$

$$\int \frac{dA}{A - \frac{\beta}{\alpha}} = -\int \alpha \, dt \Rightarrow \ln\left|A - \frac{\beta}{\alpha}\right| = -\alpha t + C$$

$$\left|A - \frac{\beta}{\alpha}\right| = Ce^{-\alpha t} \Rightarrow A = \frac{\beta}{\alpha} + Ce^{-\alpha t}$$

$$A(0) = \frac{\beta}{\alpha} + C = A_0$$
$$\Rightarrow C = A_0 - \frac{\beta}{\alpha}$$

Hence $A(t) = \frac{\beta}{\alpha} + \left(A_0 - \frac{\beta}{\alpha}\right)e^{-\alpha t}$

where $\alpha = k_1 + k_{-1}$

$\beta = k_{-1}(A_0 + B_0)$.