Math 413.
Homework 3 Solutions.

#2.11 $\varepsilon x^4 - x - 1 = 0$

Regular perturbation: $x \sim x_0 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 + \ldots$

$$\varepsilon(x_0 + \varepsilon^\alpha x_1 + \ldots)^4 - (x_0 + \varepsilon^\alpha x_1 + \ldots) - 1 = 0$$

$O(1)$: $x_0 + 1 = 0 \quad x_0 = -1$

$O(\varepsilon)$: $\varepsilon x_0^4 - \varepsilon^\alpha x_1 = 0 \Rightarrow \alpha = 1$

$x_0^4 - x_1 = 0 \Rightarrow x_1 = 1$

Hence $x \sim -1 + \varepsilon + \ldots$

This approximation only works for one of the roots. The other needs to be found by singular perturbation (as suggested by the fact that it goes to ∞ as $\varepsilon \to 0$).

Singular perturbation: $\tilde{x} = \frac{x}{\varepsilon^\gamma}$

$$\varepsilon^{1+4\gamma} \tilde{x}^4 - \varepsilon^{\gamma} \tilde{x} - 1 = 0$$

$O(\varepsilon^{1+4\gamma}) = O(\varepsilon^\gamma) \Rightarrow 1+4\gamma = \gamma \Rightarrow \gamma = -\frac{1}{3} \Rightarrow \boxed{x = \varepsilon^{-\frac{1}{3}} \cdot \tilde{x}}$

$O(\varepsilon^{-\frac{1}{3}}) < O(1)$ so this γ value works.

$$\Rightarrow \quad \tilde{x}^4 - \tilde{x} - \varepsilon^{\frac{1}{3}} = 0$$

$$\tilde{x} = \tilde{x}_0 + \varepsilon^\alpha \tilde{x}_1 + \varepsilon^\beta \tilde{x}_2 \Rightarrow$$

$$(\tilde{x}_0 + \varepsilon^\alpha \tilde{x}_1 + \ldots)^4 - (\tilde{x}_0 + \varepsilon^\alpha \tilde{x}_1 + \ldots) - \varepsilon^{\frac{1}{3}} = 0$$

$O(1)$: $\tilde{x}_0^4 - \tilde{x}_0 = 0 \quad \tilde{x}_0(\tilde{x}_0^3 - 1) = 0$

$\tilde{x}_0 = 0$ OR $\tilde{x}_0 = 1$

$O(\varepsilon^{\frac{1}{3}})$: $4\tilde{x}_0 \tilde{x}_1 \cdot \varepsilon^\alpha - \tilde{x}_1 \varepsilon^\alpha - \varepsilon^{\frac{1}{3}} = 0$

$\Rightarrow \alpha = \frac{1}{3}$ and $4\tilde{x}_0 \tilde{x}_1 - \tilde{x}_1 - 1 = 0$

Case 1: $\tilde{x}_0 = 0$, $\tilde{x}_1 = -1 \Rightarrow \tilde{x} = -\varepsilon^{\frac{1}{3}} + \ldots$

$\boxed{x = -1 + \varepsilon + \ldots}$ Same as regular perturbation soln

Case 2:

$\tilde{x}_0 = 1 \quad \tilde{x}_1 = \frac{1}{3}$

$\Rightarrow \tilde{x} = 1 + \frac{1}{3}\varepsilon^{\frac{1}{3}} + \ldots$

$$\boxed{x = \varepsilon^{-\frac{1}{3}}\left(1 + \frac{1}{3}\varepsilon^{\frac{1}{3}} + \ldots\right)}$$

#2.17 $\begin{cases} \varepsilon y'' = a - y' \\ y(0)=0 \\ y(1)=1 \end{cases}$ $\quad 0<x<1$, $\quad a>0.$

(a) BL @ $x=0$:

Outer solution: $y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2$

$$\varepsilon(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \ldots) = a - y_0' - \varepsilon y_1' - \varepsilon^2 y_2' - \ldots$$

$$O(1): \begin{cases} y_0' = a \\ y_0(1) = 1 \end{cases} \Rightarrow \quad y_0 = ax + b, \quad y_0(1) = a + b = 1 \Rightarrow b = 1-a$$

$$\Rightarrow \boxed{y_0^{(1)} = ax + 1 - a}$$

Inner solution: $\tilde{x} = \frac{x}{\varepsilon^\delta} \Rightarrow \varepsilon^{1-2\delta} y'' = a - \varepsilon^{-\delta} y'$

$$O(\varepsilon^{1-2\delta}) = O(\varepsilon^{-\delta}) \Rightarrow \delta = 1$$

$$\Rightarrow \boxed{\tilde{x} = \frac{x}{\varepsilon}}$$

$$y'' + y' - \varepsilon a = 0.$$

$O(1)$: $y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2$

$$\Rightarrow \begin{cases} y_0'' + y_0' = 0 \\ y_0(0) = 0 \end{cases} \Rightarrow \quad r^2 + r = 0, \quad y_0 = C_1 + C_2 e^{-\tilde{x}}, \quad y_0(0) = C_1 + C_2 = 0$$

$$\Rightarrow \boxed{y_0^{(2)} = C_1 - C_1 e^{-\tilde{x}}}$$

Matching $C_1 = \lim_{x\to\infty} y_0^{(2)} = \lim_{x\to 0} y_0^{(1)} = 1-a$

$$\Rightarrow \quad y_0^{(2)} = 1 - a - (1-a)e^{-x/\varepsilon}$$

So composite solution looks like

$$y = y_0^{(1)} + y_0^{(2)} - y_0^{(1)}(0) =$$

$$= ax + (1-a) + 1 - a - (1-a)e^{-x/\varepsilon} - (1-a)$$

$$\Rightarrow \boxed{y = ax + (1-a)(1-e^{-x/\varepsilon})}$$

(b) If $a=1$, composite solution gives $\boxed{y = x}$

Exact solution: $\varepsilon y'' + y' = a \qquad y_p = ax$

$$\varepsilon r^2 + r = 0$$

$$(\varepsilon r + 1) r = 0 \Rightarrow y = C_1 + C_2 e^{-x/\varepsilon} + ax$$

$$\begin{cases} y(0) = C_1 + C_2 = 0 \\ y(1) = C_1 + C_2 e^{-1/\varepsilon} + a = 1 \end{cases} \xRightarrow{a=1} \begin{cases} C_1 + C_2 = 0 \\ C_1 + C_2 e^{-1/\varepsilon} = 0 \end{cases} \Rightarrow C_1 = C_2 = 0$$

So exact solution $y = x$ matches composite.

#2.17 (cont.)

(c) BL @ $x=1$

<u>outer</u>: $y_0^{(1)} = ax + b$

$y_0^{(1)}(0) = 0 \Rightarrow b = 0$, $\boxed{y_0^{(1)} = ax}$

<u>inner</u>: $\tilde{x} \sim \frac{x-1}{\varepsilon^{\delta}}$

Since equation does not depend on x explicitly, the form of the inner solution remains the same!

$y_0^{(2)} = C_1 + C_2 e^{-\tilde{x}}$, $x = 1 \Rightarrow \tilde{x} = 0$

$y_0^{(2)}(1) = 1 \Rightarrow C_1 + C_2 e^{0} = 1$

$C_1 + C_2 = 1$

$\Rightarrow \boxed{y_0^{(2)} = 1 - C_2 + C_2 e^{-\tilde{x}}} = 1 - C_2 + C_2 e^{-\frac{x-1}{\varepsilon}}$

Matching: $a = \lim_{x \to 1} y_0^{(1)} = \lim_{x \to -\infty} y_0^{(2)} = 1 - C_2 + C_2 \cdot \infty$

This means that $C_2 = 0$ and $a = 1$ are the only choices of parameters for which this procedure works $\Rightarrow$ not possible in general.

#2.23 $\begin{cases} \varepsilon y'' - 3y' - y^4 = 0 \\ y(0) = 1,\ y(1) = 1 \end{cases}$ BL @ $x=1$

<u>Outer</u>: $y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots$

$\Rightarrow \boxed{-3y_0' - y_0^4 = 0,\ y_0(0) = 1}$

$3\frac{dy_0}{dx} = -y_0^4 \Rightarrow \frac{dy_0}{dx} = -\frac{1}{3}y_0^4$

$\frac{dy_0}{y_0^4} = -\frac{1}{3}dx$

$-\frac{1}{3}y_0^{-3} = -\frac{1}{3}x + c$

$y_0^{-3} = x + C$

$\Rightarrow y_0 = \frac{1}{\sqrt[3]{x+C}}$

$y_0(0) = 1 \Rightarrow C = 1$

$\boxed{y_0^{(1)}(x) = \frac{1}{\sqrt[3]{1+x}}}$

Inner: $\tilde{x} = \frac{x-1}{\varepsilon^{\delta}}$, $x = 1+\varepsilon^{\delta}\tilde{x}$

$$\varepsilon^{1-2\delta} y'' - 3\varepsilon^{-\delta} y' - y^4 = 0$$

$$1-2\delta = -\delta \Rightarrow \delta = 1 \Rightarrow \boxed{x = 1+\varepsilon\tilde{x}}$$

$$y'' - 3y' - \varepsilon y^4 = 0$$

$$y'' - 3y' = \varepsilon y^4, \quad y \approx y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots$$

$O(1)$: $y_0'' - 3y_0' = 0$ $\qquad r^2 - 3r = 0$

$y_0(0) = 1$ $\qquad r(r-3) = 0$

$x = 1 \Rightarrow \tilde{x} = 0$ $\qquad y_0 = C_1 + C_2 e^{3\tilde{x}}$

$y_0(0) = C_1 + C_2 = 1$

$\Rightarrow C_2 = 1 - C_1$

$$y_0^{(2)}(x) = C_1 + (1-C_1)e^{\frac{3(x-1)}{\varepsilon}}$$

Matching: $\frac{1}{2^{\frac{1}{3}}} = \lim_{x\to 1} y_0^{(1)} = \lim_{x\to -\infty} y_0^{(2)} = C_1$

$$\Rightarrow y_0^{(2)} = 2^{-\frac{1}{3}} + (1-2^{-\frac{1}{3}})e^{\frac{3(x-1)}{\varepsilon}}$$

Composite solution: $y = y_0^{(1)} + y_0^{(2)} - y_0^{(1)}(1)$

$$\Rightarrow y = \frac{1}{\sqrt[3]{1+x}} + \cancel{2^{-\frac{1}{3}}} + (1-2^{-\frac{1}{3}})e^{\frac{3(x-1)}{\varepsilon}} - \cancel{2^{-\frac{1}{3}}}$$

$$\boxed{y = \frac{1}{\sqrt[3]{1+x}} + (1-2^{-\frac{1}{3}})e^{\frac{3(x-1)}{\varepsilon}}}$$

Part II. $\begin{cases} \varepsilon y'' + y' = 2x \\ y(0) = 1, \; y(1) = 1 \end{cases}$

Outer: $y_0' = 2x$ $\qquad (y \sim y_0 + \varepsilon y_1 + \dots)$

$y_0 = x^2 + C$

$y_0(1) = 1 = 1 + C \Rightarrow C = 0$, $\boxed{y_0^{(1)} = x^2}$

Inner: $\tilde{x} = \frac{x}{\varepsilon^{\delta}} \Rightarrow \varepsilon^{1-2\delta} y'' + \varepsilon^{-\delta} y' = 2\varepsilon^{\delta}\tilde{x}$

$1-2\delta = -\delta \Rightarrow \boxed{\delta = 1}$ $\quad O(\varepsilon^{-1}) < O(\varepsilon)$

$$\Rightarrow y'' + y' = 2\varepsilon^2 \tilde{x}$$

$O(1)$: $y_0'' + y_0' = 0$

$y_0 = C_1 + C_2 e^{-\tilde{x}}$

$y_0(0) = C_1 + C_2 = 1 \Rightarrow C_1 = 1 - C_2$

$\boxed{y_0^{(2)} = 1 - C_2 + C_2 e^{-x/\varepsilon}}$

Matching: $0 = \lim_{x\to 0} y_0^{(1)} = \lim_{x\to\infty} y_0^{(2)} = 1 - C_2 \Rightarrow C_2 = 1$, $\boxed{y = x^2 + e^{-x/\varepsilon}}$