Math 413. Fall '10

Homework 2 Solutions

$\boxed{1.12}$. $\tau \frac{\partial^2 u}{\partial x^2} + \mu u = p$, $0<x<\ell$, $u(0)=0$, $u(\ell)=U$.

$u(x)$ - displacement , $[u]=L$

$[p] = \frac{MLT^{-2}}{L} = MT^{-2}$

(a) $\underbrace{[\tau]\cdot\frac{L}{L^2}}_{[\frac{\partial^2 u}{\partial x^2}]} + [\mu]\cdot \underbrace{L}_{[u]} = \underbrace{MT^{-2}}_{[p]}$ $\Rightarrow$ $[\tau] = LMT^{-2}$, $[\mu] = L^{-1}MT^{-2}$

all terms have to have same dimensions.

(b) Introduce new variables $x = x_c \cdot s$, $u = u_c \cdot v$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \frac{1}{x_c^2}\frac{\partial^2}{\partial s^2} \text{ , so that}$$

$$\frac{\tau}{x_c^2}\frac{\partial^2(u_c v)}{\partial s^2} + \mu\cdot u_c \cdot v = p \Rightarrow$$

$$\frac{\partial^2 v}{\partial s^2} + \left(\frac{\mu x_c^2}{\tau}\right)v = \frac{p x_c^2}{\tau u_c}$$

$$\left[\frac{\mu x_c^2}{\tau}\right] = \frac{L^{-1}MT^{-2}L^2}{LMT^{-2}} = 1$$

$$\left[\frac{p x_c^2}{\tau u_c}\right] = \frac{MT^{-2}L^2}{LMT^{-2}L} = 1$$

$\Rightarrow$ both groups are dimensionless.

Boundary Conditions:

$$0 = u(0) = u_c \cdot v(0) \Rightarrow v(0) = 0$$

$$U = u(\ell) = u_c \cdot v(\ell) \Rightarrow v(\ell) = \frac{U}{u_c}$$

Hence we have the non-dimensionalized problem

$$\begin{cases} \frac{\partial^2 v}{\partial s^2} + \alpha v = \beta \\ v(0) = 0 \\ v(\frac{\ell}{x_c}) = \frac{U}{u_c} \end{cases}, \quad \alpha = \frac{\mu x_c^2}{\tau}, \quad \beta = \frac{p x_c^2}{\tau u_c}$$

To solve for x_c, u_c, notice there are 4 dimensionless products:

$$\Pi_1 = \alpha, \quad \Pi_2 = \beta, \quad \Pi_3 = \frac{\ell}{x_c}, \quad \Pi_4 = \frac{U}{u_c}$$

all are independent. (over)

$\boxed{1.12}$ (cont.) By the usual rule of thumb, we put to 1 the products involved in boundary conditions first, i.e. $\Pi_3 = \frac{\ell}{x_c} = 1$

$\Pi_4 = \frac{U}{u_c} = 1 \Rightarrow \begin{array}{l} x_c = \ell \\ u_c = U \end{array}$

This transforms the problem into:

$$\begin{cases} \frac{\partial^2 v}{\partial s^2} + \alpha v = \beta, & 0 < s < 1 \\ v(0) = 0 \\ v(1) = 1 \end{cases} \qquad \text{where } \alpha = \frac{\mu \ell^2}{\tau}, \quad \beta = \frac{p \ell^2}{\tau U}$$

$\boxed{1.19}$ $\Pi = A^a B^b C^c$ is dimensionless.

(a) $f(\Pi)$ = linear in $A \Rightarrow \frac{\partial f}{\partial A}$ = const indep. of $A \Rightarrow$

$f(\Pi) = \kappa(B,C) \cdot A + \beta(B,C)$

$f(A^a B^b C^c) \Rightarrow \kappa(B,C) \cdot A = \varphi_1(A^a B^b C^c) \Rightarrow$

$\varphi_1(x) = \alpha x^{\frac{1}{a}}$

$\beta(B,C) = \varphi_2(A^a B^b C^c) \Rightarrow \varphi_2(x) = \beta$

Hence $f(\Pi) = \alpha \Pi^{\frac{1}{a}} + \beta$

(b) Now consider the case $\sqrt{AB} f(\Pi) = \kappa(B,C) A + \beta(B,C)$

$\Rightarrow f(\Pi) = \frac{\kappa(B,C)}{B^{\frac{1}{2}}} \cdot A^{\frac{1}{2}} + \frac{\beta(B,C)}{A^{\frac{1}{2}} B^{\frac{1}{2}}}$

$\Pi = A^a B^b C^c \Rightarrow \kappa(B,C) B^{-\frac{1}{2}} A^{\frac{1}{2}} = \varphi_1(A^a B^b C^c)$

$\Rightarrow \varphi_1(x) = \alpha x^{\frac{1}{2a}}$

$\beta(B,C) A^{-\frac{1}{2}} B^{-\frac{1}{2}} = \varphi_2(A^a B^b C^c)$

$\Rightarrow \varphi_2(x) = \beta x^{\frac{-1}{2a}}$

Hence $f(\Pi) = \alpha \Pi^{\frac{1}{2a}} + \beta \Pi^{-\frac{1}{2a}}$

To check this, compute $\sqrt{AB} f(\Pi) = \alpha A^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{b}{2a}} C^{\frac{c}{2a}}$

$+ \beta A^{\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}} B^{-\frac{b}{2a}} C^{-\frac{c}{2a}}$

linear in $A \rightarrow$ $= \alpha A B^{\frac{1}{2} + \frac{b}{2a}} C^{\frac{c}{2a}}$

indep. of $A \rightarrow$ $+ \beta B^{\frac{1}{2} - \frac{b}{2a}} C^{-\frac{c}{2a}}$

$\Rightarrow$ OK.

$\boxed{1.19}$ (c) $f(\Pi)=f(A^a B^b C^c)$

Suppose $f((2A)^a B^b C^c) = 4f(A^a B^b C^c)$

$\Rightarrow f(2^a(A^a B^b C^c)) = 4f(A^a B^b C^c)$

In other words, for this function $f(2^a x) = 4f(x)$

By taking derivative wrt x, $2^a \frac{df}{dx} = 4\frac{df}{dx}$

$\Rightarrow 2^a = 4 \Rightarrow \underline{a=2}$.

$\boxed{1.24}$
$$\begin{cases} u_t + Du_{xxxx} = 0 \\ u(x=0) = u_0 \\ u \to 0 \text{ as } x \to \infty \\ u(t=0) = 0 \end{cases}$$

Notice that $[u_t] = \frac{[u]}{T}$, $[u_{xxxx}] = \frac{[u]}{L^4}$, so that

$\frac{[u]}{T} + [D]\cdot\frac{[u]}{L^4} = 0$ leads to $[D] = \frac{L^4}{T}$.

Assume $u = f(x, t, D, u_0)$

$[u] = [x^a t^b D^c (u_0)^d] \Rightarrow$

$[u] = L^a T^b \left(\frac{L^4}{T}\right)^c [u]^d = L^{a+4c} T^{b-c} [u]^d$

Regardless of what $[u]$ represents, we got

$$\begin{cases} d = 1 \\ b - c = 0 \\ a + 4c = 0 \end{cases} \Rightarrow \text{ we have a choice of a free variable}$$

(a) if we choose $\begin{matrix} a = -4c \\ b = c \end{matrix} \Rightarrow u = x^{-4c} t^c D^c \cdot u_0$

So x is in the denominator

$\Rightarrow$ difficult to differentiate

(b) choose $b = c = -\frac{a}{4} \Rightarrow u = x^a t^{-\frac{a}{4}} D^{-\frac{a}{4}} \cdot u_0$

$$= \left(\frac{x}{\sqrt[4]{Dt}}\right)^a u_0 = u_0 F(\eta)$$

Where $\eta = \frac{x}{\sqrt[4]{Dt}}$ is the similarity variable.

Notice that $\frac{\partial u}{\partial t} = u_0 F'(\eta)\cdot\left(\frac{-x}{4D^{\frac{1}{4}} t^{\frac{5}{4}}}\right)$

$$\frac{\partial^2 u}{\partial x^4} = u_0 F^{IV}(\eta)\cdot\frac{1}{(D^{\frac{1}{4}} t^{\frac{1}{4}})^4} = \frac{u_0 F^{IV}(\eta)}{Dt}$$

(over)

$\boxed{1.24}$ (cont.)

$$u_t + D u_{xxxx} = 0 \Rightarrow \frac{-x F'(\eta)\cancel{u_0}}{4 D^{\frac14} t^{\frac54}} + \frac{\cancel{D} F^{IV}(\eta)\cdot \cancel{u_0}}{\cancel{D} t} = 0.$$

$$\frac{F^{IV}(\eta)}{\cancel{t}} - \frac{x}{4\sqrt[4]{Dt}\cdot\cancel{t}} F'(\eta) = 0$$

$$\frac{x}{\sqrt[4]{Dt}} = \eta$$

$$\Rightarrow F^{IV}(\eta) - \frac{\eta}{4} F'(\eta) = 0.$$

Boundary conditions: $u(x=0) = u_0 F(0) = u_0 \Rightarrow \underline{F(0)=0}.$

$$\eta = \frac{x}{\sqrt[4]{Dt}} \Rightarrow x=0 \Leftrightarrow \eta=0$$

$$x\to\infty \Leftrightarrow \eta\to\infty.$$

so $u = u_0 F(\eta) \to 0$ as $\eta\to\infty$

$\underline{F\to 0 \text{ as } \eta\to\infty.}$

Initial condition: $u(t=0) = u_0 F(\eta=\infty) = 0 \Rightarrow F\to 0$ as $\eta\to\infty$

$t=0 \Leftrightarrow \eta\to\infty$ same as above.

Hence $\begin{cases} F^{IV}(\eta) - \frac{\eta}{4} F'(\eta) = 0 \\ F(0)=0 \\ F(\infty)=0 \end{cases}$ is the resulting ODE boundary value problem.

$\boxed{2.1}$ (a) $e^{\sin\varepsilon} \approx 1 + \left(\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!}\right) + \frac{\left(\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!}\right)^2}{2} + \ldots = \underline{1 + \varepsilon + \frac{\varepsilon^2}{2} + \ldots}$

(b) $\sqrt{1+\cos\varepsilon}$

$$\approx \sqrt{1 + \underbrace{\left(1 - \tfrac12\varepsilon^2 + \tfrac{1}{4!}\varepsilon^4 + \ldots\right)}_{\cos\varepsilon}} = \sqrt{2 - \tfrac12\varepsilon^2\left(1 - \tfrac{1}{12}\varepsilon^2 + \ldots\right)} \approx$$

Since $(a+x)^{\frac12} = a^{\frac12} + \frac12 x a^{-\frac12} - \frac18 x^2 a^{-\frac32} + \ldots \Rightarrow$ $a = 2$, $x = \cancel{1} - \frac12\varepsilon^2 + \frac{1}{4!}\varepsilon^4 + \ldots$

$$\approx \sqrt2 + \frac{1}{2\sqrt2}\left(-\tfrac12\varepsilon^2 + \tfrac{1}{4!}\varepsilon^4 + \ldots\right) \approx \sqrt2 - \frac{\varepsilon^2}{4\sqrt2} = \underline{\sqrt2\left(1 - \frac{\varepsilon^2}{8}\right) + \ldots}$$

$\boxed{2.3}$ $x^2 + (1-4\varepsilon)x - \sqrt{1+4\varepsilon} = 0$. (2 roots expected)

Regular expansion: $x(\varepsilon) \sim x_0 + x_1\varepsilon^\alpha + x_2\varepsilon^\beta + \ldots$

$$\Rightarrow (x_0 + x_1\varepsilon^\alpha + x_2\varepsilon^\beta + \ldots)^2 + (1-4\varepsilon)(x_0 + x_1\varepsilon^\alpha + x_2\varepsilon^\beta) \underbrace{-1 - 2\varepsilon + 2\varepsilon^2}_{\text{exp. of } \sqrt{1+4\varepsilon}} = 0$$

(over)

$\boxed{2.3}$ (cont.)

$O(1)$: $x_0^2 + x_0 - 1 = 0 \Rightarrow x_0 = \frac{-1 \pm \sqrt{5}}{2}$

$O(\varepsilon)$: $-2\varepsilon + 2x_0x_1\varepsilon - 4x_0\varepsilon + x_1\varepsilon = 0 \qquad \alpha = 1$

$x_1(1+2x_0) - 2(1+2x_0) = 0 \Rightarrow x_1 = 2$

$O(\varepsilon^2)$ If $\beta > 1$ but $\beta \neq 2 \Rightarrow 2\varepsilon^2 - 4x_1\varepsilon^2 + x_1^2\varepsilon^2 = 0$

but $x_1 = 2 \Rightarrow 2 - 8 + 4 \neq 0$.

terms do not balance $\Rightarrow$ need to have $\beta = 2$.

Then $2\varepsilon^2 - 4x_1\varepsilon^2 + x_2\varepsilon^2 + x_1^2\varepsilon^2 + 2x_0x_2\varepsilon^2 = 0$

$-2 + x_2(1+2x_0) = 0 \Rightarrow x_2 = \frac{2}{1+2x_0} = \pm\frac{2}{\sqrt{5}}$

Hence $x \sim \frac{-1\pm\sqrt{5}}{2} + 2\varepsilon \pm \frac{2}{\sqrt{5}}\varepsilon^2$ where + corresponds to + and − to −, giving 2 roots as expected.

$\boxed{2.6}$ $x^3 = \varepsilon e^{-x}$ (one root expected)

Taking regular expansion, $x \sim x_0 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 + \ldots$

$(x_0 + x_1\varepsilon^\alpha + x_2\varepsilon^\beta)^3 = \varepsilon(1 - (x_0 + x_1\varepsilon^\alpha + x_2\varepsilon^\beta) + \frac{1}{2}(x_0 + x_1\varepsilon^\alpha + x_2\varepsilon^\beta)^2 + \ldots)$

$O(1)$: $x_0^3 = 0 \Rightarrow x_0 = 0$.

$O(\varepsilon)$: $(x_1\varepsilon^\alpha + x_2\varepsilon^\beta)^3 = \varepsilon(1 - (x_1\varepsilon^\alpha + x_2\varepsilon^\beta) + \frac{1}{2}(x_1\varepsilon^\alpha + x_2\varepsilon^\beta + \ldots)^2 \ldots)$

$x_1^3\varepsilon^{3\alpha} = \varepsilon \Rightarrow \alpha = \frac{1}{3}\ x_1 = 1$

Now we have:

$(\varepsilon^{\frac{1}{3}} + x_2\varepsilon^\beta + \ldots)^3 = \underline{\varepsilon}(1 - (\underline{\varepsilon^{\frac{1}{3}}} + x_2\varepsilon^\beta) + \frac{1}{2}(\varepsilon^{\frac{1}{3}} + x_2\varepsilon^\beta + \ldots)^2 \ldots)$

The lowest order term (after $O(\varepsilon)$) is $O(\varepsilon^{\frac{4}{3}})$ so we look for terms to balance $-\varepsilon^{\frac{4}{3}}$ with.

Candidates: $\cancel{\varepsilon} + 3\varepsilon^{\frac{2}{3}+\beta}x_2 + 3\varepsilon^{\frac{1}{3}+2\beta}x_2^2 + x_2^3\varepsilon^{3\beta} = \cancel{\varepsilon} - \varepsilon^{\frac{4}{3}} - x_2\varepsilon^{1+\beta} + \cancel{\frac{1}{2}\varepsilon^{\frac{2}{3}+1}} + \frac{1}{2}\varepsilon^{\beta+\frac{1}{3}+1}x_2$ h.o.t.

$\frac{2}{3} + \beta = \frac{4}{3} \Rightarrow \boxed{\beta = \frac{2}{3}}$

$\frac{1}{3} + 2\beta = \frac{4}{3} \Rightarrow \beta = \frac{1}{2} \Rightarrow 3x_2^2 = -1 \Rightarrow$ no sol.

$3\beta = \frac{4}{3} \Rightarrow \beta = \frac{4}{9} \Rightarrow x_2^3 = -1$, but $\varepsilon^{\frac{2}{3}+\beta}$ & $\varepsilon^{\frac{1}{3}+2\beta}$ cannot be matched with rhs $\Rightarrow$ no sol.

$\frac{4}{3} + \beta = \frac{4}{3} \Rightarrow \beta = 0$.

$1 + \beta = \frac{4}{3} \Rightarrow \beta = \frac{1}{3} \Rightarrow$ contradicts $\alpha = \frac{1}{3} < \beta$

For $\beta = \frac{2}{3}$ we have $3x_2 = -1 \Rightarrow x_2 = -\frac{1}{3} \Rightarrow \boxed{x \sim \varepsilon^{\frac{1}{3}} - \frac{1}{3}e^{\frac{2}{3}}}$

Part III. $x^2-4=\varepsilon \ln(x)$

It is simpler to treat the equation as $e^{\frac{x^2-4}{\varepsilon}}=x$

With regular expansion

$x = x_0 + x_1\varepsilon^\alpha + x_2\varepsilon^\beta + \dots$

$$\frac{x^2-4}{\varepsilon} = \frac{(x_0+x_1\varepsilon^\alpha+x_2\varepsilon^\beta)^2-4}{\varepsilon}, \quad e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots$$

$$\Rightarrow 1+\frac{(x_0+x_1\varepsilon^\alpha+x_2\varepsilon^\beta)^2-4}{\varepsilon}+\frac{((x_0+x_1\varepsilon^\alpha+x_2\varepsilon^\beta)^2-4)^2}{2\varepsilon^2}+\dots = x_0+x_1\varepsilon^\alpha + x_2\varepsilon^\beta$$

$O(1)$: $$2\varepsilon^2 + 2\varepsilon((x_0+x_1\varepsilon^\alpha+x_2\varepsilon^\beta)^2-4)+((x_0+x_1\varepsilon^\alpha+x_2\varepsilon^\beta)^2-4)^2 = 2\varepsilon^2x_0+2x_1\varepsilon^{\alpha+2}+2x_2\varepsilon^{\beta+2}$$

Constant terms: $(x_0^2-4)^2=0$

$x_0^2=4 \quad x_0=\pm 2$

$O(\varepsilon)$: $2\varepsilon(x_0^2-4)+0=0 \Rightarrow$ nothing interesting

$O(\varepsilon^2)$ $2\varepsilon^2+2\varepsilon\cdot(2x_0x_1\varepsilon^\alpha) = 2x_0\varepsilon^2$

$\alpha=1 \Rightarrow 2+4x_0x_1 = 2x_0$

$$x_1 = \frac{2x_0-2}{4x_0} = \frac{x_0-1}{2x_0}$$

$O(\varepsilon^3)$: $2\varepsilon(x_1^2\varepsilon^{2\alpha})+2\varepsilon(2x_0x_2\varepsilon^\beta) = 2x_1\varepsilon^{\alpha+2}$

$\beta=2$, $\alpha=1$ $\Rightarrow$ $2x_1^2+4x_0x_2 = 2x_1$

$x_1^2+2x_0x_2 = x_1$

$$x_2 = \frac{x_1-x_1^2}{2x_0}$$

$$x_0=2 \Rightarrow x_1=\frac{1}{4}, \quad x_2 = \frac{\frac{1}{4}-\left(\frac{1}{4}\right)^2}{4} = \frac{1}{4}\left(\frac{3}{16}\right) = \frac{3}{64}$$

$$x_0=-2 \Rightarrow x_1 = \frac{-3}{-4}=\frac{3}{4}, \quad x_2 = \frac{\frac{3}{4}-\left(\frac{3}{4}\right)^2}{2\cdot\frac{3}{4}} = \frac{2}{3}\cdot\frac{3}{16}=\frac{1}{4}$$

$x^{(1)} \sim 2+\frac{1}{4}\varepsilon+\frac{3}{64}\varepsilon^2$

$x^{(2)} \sim -2+\frac{3}{4}\varepsilon+\frac{1}{4}\varepsilon^2$