Math 413. Fall 2010.
Homework #1 Solution Set

#1.1. Assume $f = F(E, \rho, \ell)$

(a) $[f] = [E^a \cdot \rho^b \cdot \ell^c]$

T^{-1}

$[E] = ML^{-1}T^{-2}$

$[\rho] = ML^{-3}$.

$[\ell] = L$

$$T^{-1} = (ML^{-1}T^{-2})^a (ML^{-3})^b L^c$$

$$\begin{aligned} T:&\quad -1 = -2a & a &= \tfrac{1}{2} \\ M:&\quad 0 = a + b & b &= -\tfrac{1}{2} \Rightarrow f = \alpha \cdot \tfrac{1}{\ell}\sqrt{\tfrac{E}{\rho}} \\ L:&\quad 0 = -a - 3b + c & c &= -1 \end{aligned}$$

(b) Suppose full model gives $f_f = \alpha \frac{1}{\ell_f}\sqrt{\frac{E_f}{\rho_f}}$
and scaled model gives $f_s = \alpha \frac{1}{\ell_s}\sqrt{\frac{E_s}{\rho_s}}$

To use scaled model, we carry out experiment with values ℓ_s, E_s, ρ_s to get f_s. Then we compute $\alpha = f_s \ell_s \sqrt{\frac{\rho_s}{E_s}}$

Since α is supposed to be the same no matter what values of ℓ, E, ρ we choose, we can reuse it in the full-scale model, as long as the values f_f, E_f, ℓ_f, ρ_f are chosen according to

$$f_f = \left(f_s \ell_s \sqrt{\frac{\rho_s}{E_s}}\right) \frac{1}{\ell_f}\sqrt{\frac{E_f}{\rho_f}} \quad \text{or}$$

$$\frac{f_s}{f_f} = \frac{\ell_f}{\ell_s}\sqrt{\frac{\rho_f E_s}{\rho_s E_f}}$$

#1.4 (a) $F = \frac{G m_1 m_2}{d^2}$

By dimensional analysis, $[F] = [G] \cdot \frac{[m_1] \cdot [m_2]}{[d]^2}$

$$[G] = \frac{[F] \cdot [d]^2}{[m_1] \cdot [m_2]} = \frac{MLT^{-2} \cdot L^2}{M^2} = M^{-1}T^{-2}L^3$$

(b) $p = f(r, m, G)$

$[p] = [r^a m^b G^c]$

$T = L^a M^b (M^{-1} T^{-2} L^3)^c$

$$\begin{aligned} T:&\quad 1 = -2c \\ L:&\quad 0 = a + 3c \\ M:&\quad 0 = b - c \end{aligned} \qquad \begin{aligned} a &= -3c = \tfrac{3}{2} \\ b &= -\tfrac{1}{2} \\ c &= -\tfrac{1}{2} \end{aligned} \qquad \Rightarrow p = \alpha\sqrt{\frac{r^3}{mG}} \quad (1)$$

(c) Eddington result: $p = \sqrt{\dfrac{3\pi}{2\gamma G \rho}}$ is the same, if one notices that $\rho = \dfrac{m}{\frac{4}{3}\pi r^3}$ and choses $\alpha = \dfrac{2\pi}{\sqrt{\gamma}}$.

(d) To compete the formula (1), we need to be able to estimate α. This can be done by measuring the period of oscillations for a star with known m & r. The Figure 1.8 is a perfect example of a data that can be used in such calculation.

If p^* is known, and m, r are available, we Compute $\alpha = p^*\sqrt{\dfrac{mG}{r^3}}$.

#1.5 (a) $N = f(U, r, \rho, \sigma)$

$[N] = [U^a r^b \rho^c \sigma^d] = \left(\frac{L}{T}\right)^a L^b \left(\frac{M}{L^3}\right)^c \left(\frac{M}{T^2}\right)^d$

dimensionless

$$\begin{aligned} L\ |&\quad 0 = a + b - 3c \\ T\ |&\quad 0 = -a - 2d \\ M\ |&\quad 0 = c + d \end{aligned}$$

Choose c as a free parameter $\Rightarrow$ $d = -c$, $a = 2c$, $b = c$

$$N = \alpha \Big(\underbrace{\frac{U^2 r \rho}{\sigma}}_{We}\Big)^c = \alpha F(We) \quad (2)$$

(b) Interms of h - intial height; U can be obtained as follows: if $v(t)$ = velocity at time t

T = time when it hits the surface

$x(t)$ = position at time t

$x(0) = h$, $x(T) = 0$.

Write down the free fall equation:

$$\frac{d^2x}{dt^2} = -g$$

$$v = \frac{dx}{dt} = -gt + C \qquad v(0) = 0 = C$$

$$v = -gt$$

$$x = -\frac{gt^2}{2} + h \qquad \text{since } x(0) = h$$

$$0 = x(T) = -\frac{gT^2}{2} + h \Rightarrow T = \sqrt{\frac{2h}{g}} \Rightarrow u = |v(T)| = \sqrt{2hg}$$

Substituting this into (2) gives

$$N = \bar{\alpha} F\left(\frac{2hgr\rho}{\sigma}\right)$$

(c) By looking at the graph, we can assume roughly

$$N = \begin{cases} \alpha_1 h, & 0 < h < 100 \\ \alpha_2 h + \beta_2, & h > 100 \end{cases}$$

where $\alpha_1 \simeq 0.2$

$\alpha_2 \simeq 0.3$ $\beta_2 \simeq 10.$

Since $N = \bar{\alpha} F\left(\frac{2hgr\rho}{\sigma}\right) \Rightarrow$ for $0 < h < 100$

$$N = \frac{2\bar{\alpha}_1 g r \rho}{\sigma} \cdot h$$

for $h > 100$

$$N = \frac{2\bar{\alpha}_2 g \rho r}{\sigma} \cdot h + \beta_2$$

Here the slope is related to parameters in the following way:

$$\text{Slope}_1 = \alpha_1 = \bar{\alpha}_1 \left(\frac{2g r_1 \rho_1}{\sigma_1}\right)$$

$$\text{Slope}_2 = \alpha_2 = \bar{\alpha}_2 \left(\frac{2g r_1 \rho_1}{\sigma_1}\right)$$

$r_1 = 3.6 \cdot 10^{-1}\, cm$

$\rho_1 = \cancel{13.5}\ 1.1014\ \frac{gm}{cm^3}$

$\sigma_1 = 50.5\ \frac{gm \cdot cm}{s^2}$

$g = 9.8 \cdot 10^2\ \frac{cm}{s^2}$

From here:

$$\bar{\alpha}_1 = \frac{\alpha_1 \sigma_1}{2 g r_1 \rho_1}, \quad \bar{\alpha}_2 = \frac{\alpha_2 \sigma_1}{2 g r_1 \rho_1}$$

$\simeq 0.0.13 = 1.3 \cdot 10^{-2}$ $\qquad \simeq 0.0.2 = 2 \cdot 10^{-2}$

Hence the formula for N can be written as

$$N \approx \begin{cases} 0.013\left(\frac{2hgr\rho}{\sigma}\right), & 0 < h < 100 \\ 0.02\left(\frac{2hgr\rho}{\sigma}\right) + 10, & h > 100 \end{cases}$$

(d) To produce at least $N=80$, we need (in experiment 1 with r_1, ρ_1, σ_1)

$$\alpha_2 h + \beta_2 \geq 80$$

$$h \geq \frac{80-\beta_2}{\alpha_2} = \frac{70}{0.3} \simeq 233\ (cm) = 2.33\ (m)$$

(e) When $h = 200\ cm$, we are inside second linear segment

$r_2 = 3.6 \cdot 10^{-1}\ cm$

$\rho_2 = 13.5\ \frac{gm}{cm^3}$

$\sigma_2 = 435\ \frac{gm \cdot cm}{s^2}$ $\quad g = 9.8 \cdot 10^2\ \frac{cm}{s^2}$

$$N \simeq 0.02 \cdot \left(\frac{2 g r_2 \rho_2}{\sigma_2}\right) \cdot 200 + 10 \approx 97.$$

#2. $R = f(E, \rho_0, t)$

$$[R] = [E^a \rho_0^b t^c] = M^{a+b} L^{2a-3b} T^{-2a+c}$$

$$\begin{array}{c|l}
M & 0 = a + b \\
L & 1 = 2a - 3b \\
T & 0 = -2a + c
\end{array}
\qquad
\begin{array}{l}
a = \frac{1}{5} \\
b = -\frac{1}{5} \\
c = \frac{2}{5}
\end{array}$$

$$R = C E^{\frac{1}{5}} \rho_0^{-\frac{1}{5}} t^{\frac{2}{5}}$$

→ First check is that the slope of $\log R$ vs. $\log t$ is $\frac{2}{5}$.

The other information can be provided by a picture of an explosion showing radius of the shock wave at a given time t. Knowing E & ρ_0 will help estimate C. Vica versa, if C is known, one can get an approximate value for the amount of energy released in the explosion.