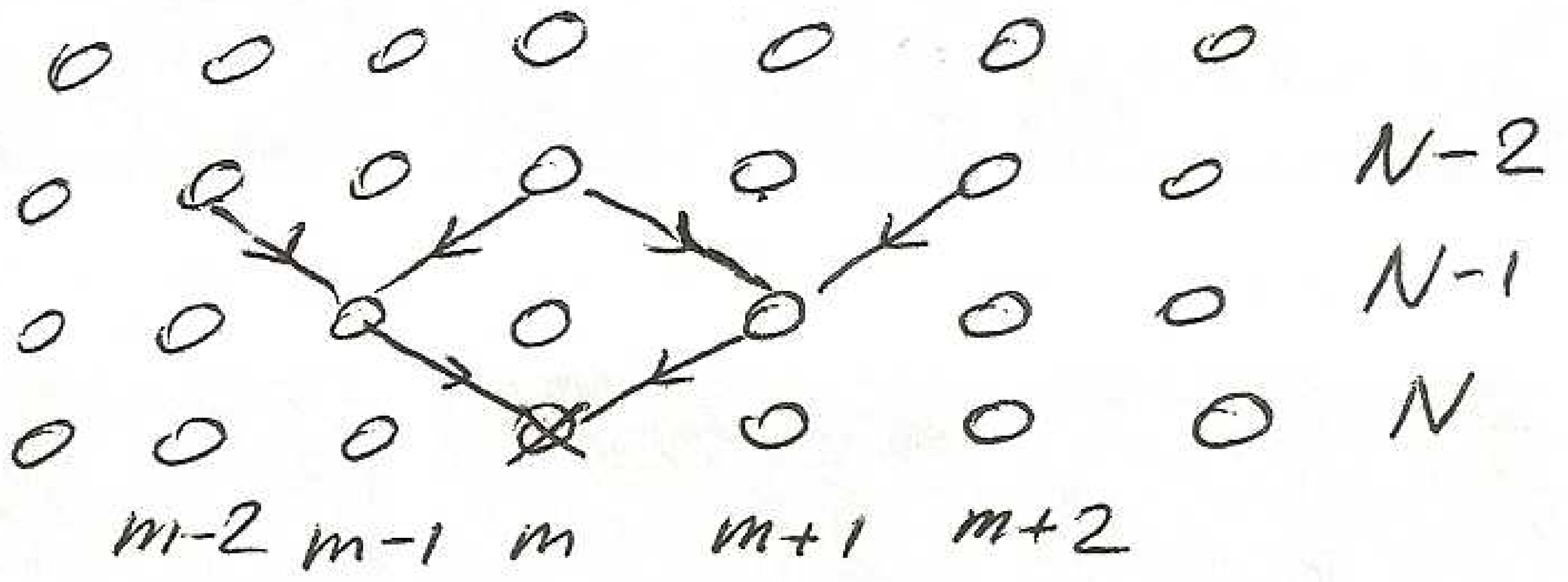
Math 413 HW5
Solution Set

#4.5

(a) $f(m,N) = \mathbb{P}\{x = m\Delta x \text{ at } t = N\Delta t \text{ from } x(t-\Delta t) = (m-1)\Delta x\}$

$g(m,N) = \mathbb{P}\{x = m\Delta x \text{ at } t = N\Delta t \text{ from } x(t-\Delta t) = (m+1)\Delta x\}$



By total probability formula:

$$
\begin{aligned}
f(m,N) &= \mathbb{P}\{x(t) = m\Delta x,\ x(t-\Delta t) = (m-1)\Delta x \mid x(t-2\Delta t) = (m-2)\Delta x\} \\
&\quad + \mathbb{P}\{x(t) = m\Delta x,\ x(t-\Delta t) = (m-1)\Delta x \mid x(t-2\Delta t) = m\Delta x\} \\
&= \mathbb{P}\{x(t-\Delta t) = (m-1)\Delta x,\ x(t-2\Delta t) = (m-2)\Delta x\} \cdot p \\
&\quad + \mathbb{P}\{x(t-\Delta t) = (m-1)\Delta x,\ x(t-2\Delta t) = m\Delta x\} \cdot (1-p) \\
&= f(m-1,N-1)\cdot p + g(m-1,N-1)\cdot(1-p)
\end{aligned}
$$

$$
\begin{aligned}
g(m,N) &= \mathbb{P}\{x(t) = m\Delta x,\ x(t-\Delta t) = (m+1)\Delta x \mid x(t-2\Delta t) = m\Delta x\} \\
&\quad + \mathbb{P}\{x(t) = m\Delta x,\ x(t-\Delta t) = (m+1)\Delta x \mid x(t-2\Delta t) = (m+2)\Delta x\} \\
&= \mathbb{P}\{x(t-\Delta t) = (m+1)\Delta x,\ x(t-2\Delta t) = m\Delta x\}(1-p) \\
&\quad + \mathbb{P}\{x(t-\Delta t) = (m+1)\Delta x,\ x(t-2\Delta t) = (m+2)\Delta x\}\, p \\
&= f(m+1,N-1)\cdot(1-p) + g(m+1,N-1)\cdot p
\end{aligned}
$$

(b) $u(x,t) = f(m,N) + g(m,N)$

$v(x,t) = f(m,N) - g(m,N)$

Note: $f = \frac{u+v}{2}$, $g = \frac{u-v}{2}$.

For simplicity of notations, let's use

$u_i^t = u(x,t)$, $u_{i-1}^t = u(x-\Delta x, t)$, $u_{i+1}^t = u(x+\Delta x, t)$

$u_i^{t-1} = u(x, t-1)$ etc ($v_i^t = v(x,t)$ etc).

So that
$$
\begin{aligned}
u_i^t &= p f_{i-1}^{t-1} + (1-p) g_{i-1}^{t-1} + (1-p) f_{i+1}^{t-1} + p g_{i+1}^{t-1} \\
&= \frac{p}{2}\left(u_{i-1}^{t-1} + v_{i-1}^{t-1}\right) + \frac{1-p}{2}\left(u_{i-1}^{t-1} - v_{i-1}^{t-1}\right) \\
&\quad + \left(\frac{1-p}{2}\right)\left(u_{i+1}^{t-1} + v_{i+1}^{t-1}\right) + \frac{p}{2}\left(u_{i+1}^{t-1} - v_{i+1}^{t-1}\right)
\end{aligned}
$$

It follows that

$$\boxed{p-\tfrac{1}{2}=\frac{1-\alpha\Delta t}{2}}$$

$$u_i^t=\tfrac{1}{2}(u_{i-1}^{t-1}+u_{i+1}^{t-1})+(p-\tfrac{1}{2})(v_{i-1}^{t-1}-v_{i+1}^{t-1});$$

Similarly, $v_i^t=\frac{1}{2}(u_{i-1}^{t-1}-u_{i+1}^{t-1})+(p-\frac{1}{2})(v_{i-1}^{t-1}+v_{i+1}^{t-1})$

Expanding around (x,t) with $\Delta x,\Delta t$ - small:

$$u=\tfrac{1}{2}\Big(u-\cancel{\Delta x u_x}-\Delta t u_t+\tfrac{1}{2}(\Delta x^2 u_{xx}+\cancel{2\Delta x\Delta t\, u_{xt}}+\Delta t^2 u_{tt})$$
$$+u+\cancel{\Delta x u_x}-\Delta t u_t+\tfrac{1}{2}(\Delta x^2 u_{xx}-\cancel{2\Delta x\Delta t\, u_{xt}}+\Delta t^2 u_{tt})\Big)$$
$$+\left(\frac{1-\alpha\Delta t}{2}\right)\Big(\cancel{v}-\Delta x v_x-\cancel{\Delta t v_t}+\tfrac{1}{2}(\cancel{\Delta x^2 v_{xx}}+2\Delta x\Delta t\, v_{xt}+\cancel{\Delta t^2 v_{tt}})$$
$$-\cancel{v}-\Delta x v_x+\cancel{\Delta t v_t}-\tfrac{1}{2}(\cancel{\Delta x^2 v_{xx}}-2\Delta x\Delta t\, v_{xt}+\cancel{\Delta t^2 v_{tt}})\Big)$$

$$\Rightarrow\quad u=\tfrac{1}{2}(2u-2\Delta t u_t+\Delta x^2 u_{xx}+\Delta t^2 u_{tt})$$
$$+\left(\frac{1-\alpha\Delta t}{2}\right)(-2\Delta x v_x+2\Delta x\Delta t\, v_{xt})$$

Hence $\frac{1}{2}(-2\Delta t u_t+\Delta x^2 u_{xx}+\Delta t^2 u_{tt})+\left(\frac{1-\alpha\Delta t}{2}\right)(-2\Delta x v_x+2\Delta x\Delta t\, v_{xt})=0$ $\boxed{(1)}$

Following the same argument for v_i^t, we get

$$-\alpha\Delta t\cdot v+(-\Delta x u_x+\Delta x\Delta t\, u_{xt})+\frac{1-\alpha\Delta t}{2}(-2\Delta t v_t+\Delta x^2 v_{xx}+\Delta t^2 v_{tt})=0\quad\boxed{(2)}$$

(2) leads to $\alpha\Delta t(v-\Delta t v_t)+\Delta t v_t-\left(\frac{1-\alpha\Delta t}{2}\right)(\Delta x^2 v_{xx}+\Delta t^2 v_{tt})$
$$=-\Delta x u_x+\Delta x\Delta t\, u_{xt}$$

Differentiating wrt x:

$$\alpha\Delta t(v_x-\Delta t v_{xt})+\Delta t v_{xt}-\left(\frac{1-\alpha\Delta t}{2}\right)(\Delta x^2 v_{xxx}+\Delta t^2 v_{ttx})$$
$$=-\Delta x u_{xx}+\Delta x\Delta t\, u_{xxt}$$

$$\Rightarrow\quad v_x-\Delta t v_{xt}=-\frac{1}{\alpha}v_{xt}+\frac{1-\alpha\Delta t}{2\alpha\Delta t}(\Delta x^2 v_{xxx}+\Delta t^2 v_{ttx})$$
$$-\frac{\Delta x}{\alpha\Delta t}\cdot u_{xx}+\frac{\Delta x}{\alpha}u_{xxt}$$

Substituting this into (1):

$$0=\tfrac{1}{2}\Big(-2\Delta t u_t+\Delta x^2 u_{xx}+\frac{2\Delta x^2}{\alpha\Delta t}(1-2\Delta t)u_{xx}+\Delta t^2 u_{tt}\Big)$$
$$-\Delta x(1-2\Delta t)\cdot\Big[-\frac{1}{\alpha}v_{xt}+\frac{1-\alpha\Delta t}{2\alpha\Delta t}(\Delta x^2 v_{xxx}+\Delta t^2 v_{ttx})+\frac{\Delta x}{\alpha}u_{xxt}\Big]$$

$$\Rightarrow\quad -u_t+\frac{1}{2}\frac{\Delta x^2}{\alpha\Delta t^2}(\alpha\Delta t-4\Delta t+2)u_{xx}+\frac{\Delta t}{2}u_{tt}-\frac{\Delta x}{\Delta t}(1-2\Delta t)\cdot[\ldots]=0$$

where $[\ldots]=-\frac{1}{\alpha}v_{xt}+\frac{1-\alpha\Delta t}{2\alpha\Delta t}(\Delta x^2 v_{xxx}+\Delta t^2 v_{ttx})+\frac{\Delta x}{\alpha}u_{xxt}\xrightarrow[\Delta x\to 0]{\Delta t\to 0}-\frac{1}{\alpha}v_{xt}$

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So $-u_t + \frac{1}{2\alpha}\frac{\Delta x^2}{\Delta t^2}\cdot 2u_{xx} + \frac{\Delta t}{2}u_{tt} - \frac{\Delta x}{\Delta t}\left(-\frac{1}{\alpha}v_{xt}\right) = 0$

Notice that in 1st approximation, $u = u - \Delta t\, u_t + \frac{1-\alpha\Delta t}{2}(-2\Delta x\, v_x)$

$\Rightarrow \Delta t\, u_t = (\alpha\Delta t - 1)\Delta x\, v_x$

so that $v_{xt} = \frac{\Delta t\, u_{tt}}{\Delta x(\alpha\Delta t - 1)}$. Let $\boxed{c = \lim \frac{\Delta x}{\Delta t}}$ $\Rightarrow$

$$-u_t + \frac{1}{\cancel{2}\alpha}\left(\frac{\Delta x}{\Delta t}\right)^2\cdot u_{xx} + \frac{\Delta t}{2}u_{tt} + \frac{\cancel{\Delta x}}{\alpha\cancel{\Delta t}}\,\frac{\cancel{\Delta t}\, u_{tt}}{\cancel{\Delta x}(\alpha\Delta t - 1)} = 0$$

$$-u_t + \frac{1}{\alpha}\cdot c^2\cdot u_{xx} + \frac{\Delta t}{2}u_{tt} + \frac{u_{tt}}{\alpha(\alpha\Delta t - 1)} = 0.$$

$$-\alpha(\alpha\overset{\to 0}{\Delta t} - 1)u_t + c^2(\alpha\overset{\to 0}{\Delta t} - 1)u_{xx} + \left[\alpha(\alpha\Delta t - 1)\cdot\left(\overset{\to 0}{\frac{\Delta t}{2}}\right) + 1\right]u_{tt} = 0$$

Letting $\Delta t \to 0$: $\alpha u_t - c^2 u_{xx} + u_{tt} = 0$

$\Rightarrow \boxed{u_{tt} + \alpha u_t = c^2 u_{xx}}$ telegraph equation

(c) The derivation remains the same, so we still get ~~$u_t + \frac{1}{2}$~~ $0 = \frac{1}{2}\left(-2\Delta t\, u_t + \Delta x^2 u_{xx} + \frac{2\Delta x^2}{\alpha\Delta t}(1-2\Delta t)u_{xx} + \right.$

$\left. + \Delta t^2 u_{tt}\right) - \Delta x(1-2\Delta t)\cdot\underset{0}{[\ldots]} = 0$

Dividing by Δt and letting $\Delta x, \Delta t \to 0$:

$$0 = -u_t + \frac{1}{2}\frac{\Delta x^2}{\Delta t}u_{xx} + \frac{\Delta x^2}{\alpha\Delta t^2}(1-2\Delta t)u_{xx} + \overset{\to 0}{\frac{\Delta t}{2}u_{tt}} - \overset{\to 0}{\frac{\Delta x(1-2\Delta t)}{\Delta t}[\,]}$$

$$u_t = \frac{1}{2}\frac{\Delta x^2}{\Delta t}\left[1 + \frac{2(1-2\Delta t)}{\alpha\Delta t}\right]u_{xx}$$

$$u_t = \frac{1}{2}\frac{\Delta x^2}{\Delta t}\left[\frac{\alpha\Delta t + 2 - 4\Delta t}{\alpha\Delta t}\right]u_{xx} \qquad \alpha\Delta t = 2(1-p)$$

$$u_t = \frac{1}{2}\frac{\Delta x^2}{\Delta t}\left[\frac{2(1-p) + 2 - \overset{\to 0}{4\Delta t}}{2(1-p)}\right]u_{xx} \Rightarrow u_t = \frac{1}{2}\frac{(\Delta x)^2}{\Delta t}\left[\frac{2-p}{1-p}\right]u_{xx}$$

$u_t = Du_{xx}$, $D = \frac{1}{2}\frac{(\Delta x)^2}{\Delta t}\left(\frac{2-p}{1-p}\right)$

(differs from the book - must be a typo).

#4.18. $\begin{cases} u_t = Du_{xx} - cu & -\infty < x < \infty \\ u(x,0) = f(x) & t > 0 \end{cases}$

(a) $A \to B$ gives rise to reaction term

(b) $\mathcal{F}(u_t) = \mathcal{F}(Du_{xx}) - c\mathcal{F}(u)$, Let $U = \mathcal{F}(u)$ $\Rightarrow$

$\frac{d}{dt}U = -Dk^2U - cU = -(Dk^2 + c)U$

$U(k,0) = F(k)$

$\Rightarrow U = F(k)e^{-(Dk^2+c)t} \Rightarrow u(x,t) = \frac{e^{-ct}}{2\sqrt{\pi Dt}}\int_{-\infty}^{+\infty} f(s)e^{-\frac{(x-s)^2}{4Dt}}ds$

(c) let $u = ve^{at}$

Then $\left.\begin{aligned} u_t &= v_t e^{at} + a v e^{at} \\ u_{xx} &= v_{xx} e^{at} \end{aligned}\right\} \Rightarrow$

$v_t e^{at} + a v e^{at} = D v_{xx} e^{at} - c v e^{at}$

$\Rightarrow$ Choose $a = -c$ and divide by $e^{at} \Rightarrow$

$v_t = D v_{xx}$

We know $v(x,t) = \dfrac{1}{2\sqrt{\pi D t}} \displaystyle\int_{-\infty}^{+\infty} f(s) e^{-(x-s)^2/4Dt}\, ds$

So $u(x,t) = e^{at} v(x,t) =$

$= \dfrac{e^{-ct}}{2\sqrt{\pi D t}} \displaystyle\int_{-\infty}^{+\infty} f(s) e^{-(x-s)^2/4Dt}\, ds$ (same as above)

(d) $u_t = D u_{xx} + cu$
can be obtained, say, by a reaction $A \to 2A$ instead of $A \to B$.

Part III. $V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \quad S \geq 0, \quad 0 \leq t \leq T$

(a) $S = Ke^x$

$V(S,t) = K v(x,\tau), \quad \tau = \dfrac{(T-t)\sigma^2}{2}$

$\dfrac{\partial V}{\partial t} = K \dfrac{\partial v}{\partial \tau} \cdot \dfrac{\partial \tau}{\partial t} = -K \dfrac{\sigma^2}{2} \dfrac{\partial v}{\partial \tau}$

$\dfrac{\partial V}{\partial S} = K \dfrac{\partial v}{\partial x} \cdot \dfrac{\partial x}{\partial S} = K \dfrac{\partial v}{\partial x} \dfrac{\partial}{\partial S}\left(\ln \dfrac{S}{K}\right) = \dfrac{K}{S} \dfrac{\partial v}{\partial x}$

$\dfrac{\partial^2 V}{\partial S^2} = \dfrac{\partial}{\partial S}\left(\dfrac{K}{S}\dfrac{\partial v}{\partial x}\right) = -\dfrac{K}{S^2}\dfrac{\partial v}{\partial x} + \dfrac{K}{S}\left(\dfrac{\partial x}{\partial S} \cdot \dfrac{\partial}{\partial x}\right)\dfrac{\partial v}{\partial x} = -\dfrac{K}{S^2}\dfrac{\partial v}{\partial x} + \dfrac{K}{S^2}\dfrac{\partial^2 v}{\partial x^2}$

$\Rightarrow -K\dfrac{\sigma^2}{2}\dfrac{\partial v}{\partial \tau} + \dfrac{1}{2}\sigma^2\left(-K\dfrac{\partial v}{\partial x} + K\dfrac{\partial^2 v}{\partial x^2}\right) + rK\dfrac{\partial v}{\partial x} - rKv = 0$

$\dfrac{\partial v}{\partial \tau} = \dfrac{\partial^2 v}{\partial x^2} + \left(\dfrac{2r}{\sigma^2} - 1\right)\dfrac{\partial v}{\partial x} - \dfrac{2r}{\sigma^2} v \Rightarrow$

$v_\tau = v_{xx} + \alpha v_x + \beta v$, where $\alpha = \dfrac{2r}{\sigma^2} - 1, \ \beta = -\dfrac{2r}{\sigma^2}$

(b) Let $\mathcal{F}\{v\} = U(k)$ $\mathcal{F}\{f\} = F(k)$

$\Rightarrow \dfrac{dU}{d\tau} = -Dk^2 U + \alpha i k U + \beta U = (-Dk^2 + \alpha i k + \beta) U$

$\Rightarrow U = F(k) e^{-Dk^2\tau + \alpha i k \tau + \beta\tau} = e^{\beta\tau} F(k) e^{-Dk^2\tau - i\alpha\tau k}$

$\Rightarrow u(x,\tau) = \dfrac{e^{\beta\tau}}{2\sqrt{\pi D\tau}} \displaystyle\int_{-\infty}^{+\infty} f(s) \cdot e^{-\frac{(x-\alpha\tau-s)^2}{4D\tau}}\, ds$

$\Rightarrow V(S,t) = \dfrac{K e^{\beta\tau}}{2\sqrt{\pi D\tau}} \displaystyle\int_{-\infty}^{+\infty} f(s) e^{-\frac{(x-\alpha\tau-s)^2}{4D\tau}}\, ds$, where $\tau = \dfrac{(T-t)\sigma^2}{2}$