Math 413 Fall 2010. Homework 5.
Due Wednesday Dec 1.

You are allowed to work in groups of 2-3 people on this assignment. Please submit one report clearly indicating all members involved.

Part I.
Problems 4.5, 4.18, Holmes textbook.

Part II.
Diffusion equation in any dimension can be written as $u_t = D\nabla^2 u$. In 1-d, we can think of the following visualization of the diffusion process: a large number of atoms are initially placed at the origin of a coordinate system at time $t = 0$, then the atoms undergo random displacements of some average distance $\langle r \rangle$ at some frequency Γ (steps per time). The concentration of atoms in this case satisfies the point source solution formula:

$$u(x,t) = \frac{N}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)} \qquad (1)$$

where N is the number of diffusing atoms, which should be large enough to be able to take continuous limit. One way to approximate diffusion constant is through the formula $D = \alpha\Gamma\langle r\rangle^n$, where Γ is the hopping rate and α and n are dimensionless constants which may depend on the dimension of the diffusion problem.

(a) Simulate a random walk in one dimension by iterating a single random walk step manytimes. At each iteration, an atom will hop a distance of 1 step in the plus- or minus-x direction.Do this for enough atoms and a series of times so that you can plot distributions ofrandom walker positions at several times. You can associate the time with the numberof iterations, so that Γ will have units of (length)^n/(time per iteration).

(b) Compare your distributions to the continuum solution to the diffusion equation given above.

(c) Use your distributions to fit the data to equation (1) so that you can find estimates to the coefficents α,Γ and n for one dimensional diffusion.

(d) (optional) Repeat steps (a)-(c) for a 2-dimensional problem. Visualize your results.

Part III.
The Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, S \geq 0, 0 \leq t \leq T \qquad (2)$$

is used extensively in finance to model European-style options. Here $V(S,t)$ is the value of the option, S is the price, t is the time, T the expiration date, σ the volatility and r the risk-free interest rate.

(a) Assume r, σ are constant and show that equation (2) can be transformed into a diffusion equation by the following change of variables: $S = Ke^x, V(S,t) = Kv(x,\tau), \tau = (T-t)\sigma^2/2$.

(b) Solve the resulting diffusion equation and derive the corresponding formula for the Black-Scholes equation solution, assuming initial condition $f(x,0)$.

(c) (optional) Comment on the use of this result in financial market analysis.