Math 214. Practice Exam Solns

Q1. $f(\lambda, g) = \left[\frac{1}{T}\right]$

$\lambda \to L$, $g \to \frac{M}{T^2}$

$\frac{1}{T} = L^a \cdot \left(\frac{\cancel{M}}{T^2}\right)^b$

$T: \quad -1 = -2b$

$L: \quad 0 = a + b$

$b = \frac{1}{2}$

$a = -\frac{1}{2}$

$\Rightarrow f = \varphi(\lambda^{-\frac{1}{2}} g^{\frac{1}{2}})$

$f = \varphi\left(\sqrt{\frac{g}{\lambda}}\right)$

Q2. $\begin{cases} \frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \\ y(0) = y_0 \end{cases}$

$y = y_c \cdot u$

$t = t_c \cdot s$

$\frac{1}{t_c} \cdot \frac{d(y_c u)}{ds} = r\left(1 - \frac{y_c u}{K}\right) y_c u$

$\frac{\cancel{y_c}}{t_c} \frac{du}{ds} = r\left(1 - \frac{y_c}{K} u\right) \cancel{y_c} u$

$\begin{cases} \frac{du}{ds} = t_c r\left(1 - \frac{y_c}{K} u\right) u \\ u(0) = \frac{y_0}{y_c} \end{cases}$

$\Pi_1 = \frac{y_0}{y_c}$, $\Pi_2 = t_c r$, $\Pi_3 = \frac{y_c}{K}$

~~$\Pi_1 = 1 \Rightarrow y_c = y_0$~~

$\Pi_2 = 1$, $t_c = \frac{1}{r}$

$\Pi_3 = 1$, $y_c = K$

$\Rightarrow \begin{cases} \frac{du}{ds} = (1-u)u \\ u(0) = \frac{y_0}{K} = u_0 \end{cases}$

Q3. $\begin{cases} \frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + \mu c \\ c(x,0) = 0 \\ c(0,t) = C_0 \\ c(\ell, t) = 0 \end{cases}$

$[c] = \frac{1}{L^3}$

$\frac{1}{L^3 T} = [D] \cdot \frac{1}{L^5} = [\mu] \cdot \frac{1}{L^3}$

$[D] = \frac{L^2}{T}$, $[\mu] = \frac{1}{T}$

$\begin{cases} c = c_s u \\ t = t_s \cdot s \\ x = x_s y \end{cases} \Rightarrow \frac{1}{t_s} \frac{d(\cancel{c_s} u)}{ds} = D \frac{1}{x_s^2} \frac{d(\cancel{c_s} u)}{dy} + \mu \cdot \cancel{c_s} u$

$\frac{du}{ds} = \underbrace{\left(\frac{D t_s}{x_s^2}\right)}_{\alpha} \frac{du}{dy} + \underbrace{(\mu t_s)}_{\beta} u$

$\begin{cases} u(y,0) = 0 \\ u(0,s) = \frac{C_0}{C_s} \\ u\left(\frac{\ell}{x_s}, s\right) = 0 \end{cases}$

$\Pi_1 = \frac{\ell}{x_s} = 1 \Rightarrow x_s = \ell$

$\Pi_2 = \frac{C_0}{C_s} = 1 \Rightarrow C_s = C_0$

$\Pi_3 = \alpha = 1 \Rightarrow t_s = \frac{x_s^2}{D} = \frac{\ell^2}{D}$

Q4 $x^2-4=\varepsilon e^x$

$x = x_0 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 + \ldots$

$$(x_0 + x_1\varepsilon^\alpha + x_2\varepsilon^\beta + ..)^2 - 4 = \varepsilon\left(1 + (x_0 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 +) + \tfrac{1}{2}(x_0 + \varepsilon^\alpha x_1 + .)^2\right)$$

$O(1)$: $x_0^2 - 4 = 0$ $\quad x_0 = \pm 2$

$O(\varepsilon)$: $\alpha = 1$ $\quad 2x_0 x_1 \cancel{\varepsilon} = \varepsilon(1 + x_0 + \frac{x_0^2}{2} + ..) = \cancel{\varepsilon} e^{x_0}$

$x_1 = \dfrac{e^{x_0}}{2x_0}$ (you get $x_1 = \frac{3+x_0}{2x_0}$ if you cut the exp series).

$O(\varepsilon^2)$: $x_1^2 \varepsilon^{2\alpha} + 2x_0 x_2 \varepsilon^\beta = \varepsilon^{\alpha+1} x_1 + \varepsilon x_0 x_1 \varepsilon^\alpha$

$$x_1^2 + 2x_0 x_2 = x_1 + x_0 x_1$$

$$x_2 = \frac{x_1 + x_0 x_1 - x_1^2}{2x_0}$$

$x_0 = 2$ $\quad x_1 = \dfrac{e^2}{4}$ $\quad x_2 = \dfrac{e^2/4 + e^2/2 - \left(\frac{e^2}{4}\right)^2}{4} = \dfrac{e^2}{64}(4 + 8 - e^2) = \dfrac{e^2(12-e^2)}{64}$

$x_0 = -2$ $\quad x_1 = \dfrac{e^{-2}}{-4}$

$x_2 = \dfrac{e^{-2}}{64}(-4 + 8 - e^{-2}) = \dfrac{+e^{-2}(\cancel{4} + e^{-2})}{64}$

In the case of the approximation $e^{x_0} \sim 1 + x_0 + \frac{x_0^2}{2}$ the answer is $x_1 = \begin{cases} \frac{5}{4}, & x_0 = 2 \\ \frac{1}{4}, & x_0 = -2 \end{cases}$ $\quad x_2 = \begin{cases} \frac{35}{64}, & x_0 = 2 \\ \frac{5}{64}, & x_0 = -2 \end{cases}$

Q5. $y'' + y = 4\varepsilon (y')^2$, $y(0) = 1$, $y'(0) = 0$

$y = y_0 + \varepsilon^\alpha y_1 + \varepsilon^\beta y_2 + ..$

$$\begin{cases} y_0'' + \varepsilon^\alpha y_1'' + \varepsilon^\beta y_2'' + y_0 + \varepsilon^\alpha y_1 + \varepsilon^\beta y_2 = 4\varepsilon(y_0 + \varepsilon^\alpha y_1 + \varepsilon^\beta y_2) \times (y_0' + \varepsilon^\alpha y_1' + \varepsilon^\beta y_2')^2 \\ y_0(0) + \varepsilon^\alpha y_1(0) + \varepsilon^\beta y_2(0) = 1 \\ y_0'(0) + \varepsilon^\alpha y_1'(0) + \varepsilon^\beta y_2'(0) = 0 \end{cases}$$

$O(1)$: $\begin{cases} y_0'' + y_0 = 0 \\ y_0(0) = 1 \\ y_0'(0) = 0 \end{cases} \Rightarrow y_0 = \cos t$

$O(\varepsilon)$: $\begin{cases} y_1'' + y_1 = 4y_0(y_0')^2 = 4\cos t \sin^2 t \\ y_1(0) = 0 \\ y_1'(0) = 0 \end{cases}$