Math 413. Fall 2010.

Final Exam Solutions

(1) $\begin{cases} \varepsilon y'' + y' = 2x, & 0<x<1, \quad 0<\varepsilon \ll 1 \\ y(0)=1, \; y(1)=1 \end{cases}$

We will assume there is a boundary layer @ $x=0$ and will use singular perturbation theory to approximate the solution to this BVP to the 0st order of accuracy (1-term approximation).

→ Denote $\tilde{y}(x)$ the <u>outer solution</u> satisfying

$$\begin{cases} \varepsilon \tilde{y}'' + \tilde{y}' = 2x \\ \tilde{y}(1) = 1 \end{cases}$$

Use regular perturbation to approximate $\tilde{y}(x)$ outside of the boundary layer:

$$\tilde{y} \sim \tilde{y}_0 + \varepsilon \tilde{y}_1 + \dots$$

$O(1)$: $\varepsilon \tilde{y}_0'' + \tilde{y}_0' = 2x$ leads to $\begin{cases} \tilde{y}_0' = 2x \\ \tilde{y}_0(1) = 1 \end{cases}$

$$\Rightarrow \tilde{y}_0 = x^2 + C$$

$$\tilde{y}_0(1) = 1 + C = 1 \Rightarrow C = 0$$

So $\boxed{\tilde{y}_0(x) = x^2}$

→ Denote $Y(x)$ the inner solution, i.e. solution in the boundary layer. By singular perturbation argument, let $x \sim \varepsilon^{\delta} \bar{x}$ ⇒ after the change of variables $\underline{\varepsilon^{1-2\delta}} Y'' + \underline{\varepsilon^{-\delta}} Y' = 2\varepsilon^{\delta}\bar{x}$

if $O(\varepsilon^{1-2\delta}) = O(\varepsilon^{-\delta}) \Rightarrow \boxed{\delta = 1}$ and $O(\varepsilon^{\delta})$ is higher order ⇒ than $\varepsilon^{1-2\delta}$ & $\varepsilon^{-\delta}$ terms ⇒ all works out.

$1 - 2\delta = -\delta$

Hence for $\bar{x} = \frac{x}{\varepsilon}$ we have

$$\varepsilon^{-1} Y'' + \varepsilon^{-1} Y' = 2\varepsilon \bar{x}$$

$$\begin{cases} Y'' + Y' = 2\varepsilon^2 \bar{x} \\ Y(0) = 1 \end{cases}$$

By regular perturbation: $Y(\bar{x}) \sim Y_0 + \varepsilon Y_1 + \dots$

So $\begin{cases} y_0'' + y' = 0 \\ y_0(0) = 1 \end{cases}$ is the ~~1st order~~ 1-term approx.

$\Rightarrow y_0 = C_1 + C_2 e^{-\bar{x}}$

$y_0(0) = C_1 + C_2 = 1 \Rightarrow y_0 = C_1 + (1-C_1)e^{-\bar{x}}$

$$\boxed{y_0 = C_1 + (1-C_1)e^{-x/\varepsilon}}$$

Matching the 2 parts of the solution:

$0 = \lim_{x\to 0} \tilde{y}_0(x) = \lim_{x\to\infty} y_0(x) = C_1 \Rightarrow C_1 = 0$ and $y_0 = e^{-x/\varepsilon}$

Composite solution: $\boxed{y(x) = x^2 + e^{-x/\varepsilon}}$

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$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) & t > 0 \end{cases}$$

By Fourier transform method (in x-variable):

$$\mathcal{F}[u_{tt}] = c^2 \mathcal{F}(u_{xx})$$

$$\underset{\frac{\partial^2 F(k,t)}{\partial t^2}}{\parallel} \qquad \underset{-k^2 F(k,t)}{\parallel}$$, where $F(k,t) = \mathcal{F}(u(x,t))$

Here we assume $\int_{-\infty}^{+\infty} |u_x(x,t)|\,dx < \infty$ for $t \geq 0$

and $\begin{matrix} u(x,t) \to 0 \\ u_x(x,t) \to 0 \end{matrix}$ as $x \to \pm\infty$, for $t \geq 0$.

We get $F_{tt} + c^2 k^2 F = 0$

General solution: $F(k,t) = A(k)\cos(ckt) + B(k)\sin(ckt)$

Apply Fourier transform to initial conditions:

$$\mathcal{F}[u(x,0)] = \mathcal{F}(\varphi(x)) = \Phi(k)$$

$$\mathcal{F}(u_t(x,0)) = \mathcal{F}(\psi(x)) = \Psi(k)$$

$\Rightarrow F(k,0) = [A(k) = \Phi(k)]$

$F_t(k,t) = -A(k)(ck)\sin(ckt) + B(k)(ck)\cos(ckt)$

$F_t(k,0) = B(k)(ck) = \Psi(k) \Rightarrow \left[B(k) = \frac{1}{ck}\cdot\Psi(k)\right]$

Hence: $F(k,t) = \Phi(k)\cdot\cos(ckt) + \frac{1}{ck}\Psi(k)\cdot\sin(ckt)$

Define $\mathcal{X}(x) = \int_{-\infty}^{x} \psi(s)\,ds$, so that $\mathcal{X}'(x) = \psi(x)$

and $\underset{\Psi(k)}{\underset{\parallel\,def}{\mathcal{F}(\psi(x))}} = ik\,\underset{X(k)}{\underset{def\,\parallel}{\mathcal{F}(\mathcal{X}(x))}} = ik\,X(k)$

This gives $F(k,t) = \Phi(k)\cos(ckt) + \frac{i}{c}X(k)\sin(ckt) =$

$$= \Phi(k)\cdot\frac{e^{ickt}+e^{-ickt}}{2} + \frac{1}{2c}X(k)\left(e^{ickt}-e^{-ickt}\right)$$

$$= \frac{1}{2}\left(\Phi(k)e^{ickt} + \Phi(k)e^{-ickt}\right) + \frac{1}{2c}\left(X(k)e^{ickt} - X(k)e^{-ickt}\right)$$

Recognizing the fact that

$$\Phi(k)e^{ickt} = \mathcal{F}(\varphi(x+ct))$$
$$\Phi(k)e^{-ickt} = \mathcal{F}(\varphi(x-ct))$$
$$X(k)e^{ickt} = \mathcal{F}(\mathcal{X}(x+ct))$$
$$X(k)e^{-ickt} = \mathcal{F}(\mathcal{X}(x-ct))$$

We get: $u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) +$

$$+ \frac{1}{2c}\left(\mathcal{X}(x+ct) - \mathcal{X}(x-ct)\right)$$

$\Rightarrow$ since $\mathcal{X}(x+ct) - \mathcal{X}(x-ct) = \int_{-\infty}^{x+ct}\psi(s)\,ds - \int_{-\infty}^{x-ct}\psi(s)\,ds$

$$= \int_{x-ct}^{x+ct}\psi(s)\,ds \quad\Rightarrow$$

$$\left[u(x,t) = \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds\right] \quad \text{QED.}$$

(3)
$$\begin{cases} \dfrac{ds_1}{dt} = v_1 - k_1 s_1 x_1 + k_{-1} x_2 \\ \dfrac{ds_2}{dt} = k_2 x_2 - k_3 s_2^{\gamma} e + k_{-3} x_1 - \delta_2 s_2 \\ \dfrac{dx_1}{dt} = -k_1 s_1 x_1 + (k_{-1}+k_2)x_2 + k_3 s_2^{\gamma} e - k_{-3} x_1 \\ \dfrac{dx_2}{dt} = k_1 s_1 x_1 - (k_{-1}+k_2)x_2 \end{cases}$$

with conservation law $x_1 + x_2 + e = e_0$.

(a) Nondimensionalization using scaling:

$$\sigma_1 = \frac{k_1 s_1}{k_2 + k_{-1}}, \quad \sigma_2 = \left(\frac{k_3}{k_{-3}}\right)^{1/\gamma} s_2$$

$$u_1 = \frac{x_1}{e_0}, \quad u_2 = \frac{x_2}{e_0}, \quad t = \frac{k_2 + k_{-1}}{e_0 k_1 k_2}\tau$$

yields $\frac{d}{dt} = \frac{e_0 k_1 k_2}{k_2 + k_{-1}} \frac{d}{d\tau}$ etc $\Rightarrow$

$$\frac{e_0 \cancel{k_1} k_2}{\cancel{k_2+k_{-1}}} \cdot \frac{\cancel{k_2+k_{-1}}}{\cancel{k_1}} \left(\frac{d\sigma_1}{d\tau}\right) = \nu_1 - (k_2 + k_{-1})\sigma_1 e_0 u_1 + e_0 k_{-1} u_2$$

$$\Rightarrow \frac{d\sigma_1}{d\tau} = \underbrace{\frac{1}{k_2 e_0}\cdot \nu_1}_{\overset{\text{def}}{=}\ \nu} - \frac{k_2 + k_{-1}}{k_2}\sigma_1 u_1 + \frac{k_{-1}}{k_2} u_2$$

So that $\boxed{\frac{d\sigma_1}{d\tau} = \nu - \frac{k_2 + k_{-1}}{k_2}\sigma_1 u_1 + \frac{k_{-1}}{k_2} u_2}$, with $\boxed{\nu = \frac{\nu_1}{k_2 e_0}}$

Same way, and noticing $e = e_0 - x_1 - x_2$, we get

$$\frac{d\sigma_2}{d\tau} = \frac{k_2 + k_{-1}}{e_0 k_1 k_2}\left(\frac{k_3}{k_{-3}}\right)^{1/\gamma}\left[k_2 e_0 u_2 - k_{-3}\sigma_2^{\gamma}(e_0 - e_0 u_1 - e_0 u_2) + k_{-3} e_0 u_1 - \gamma_2 \cdot \left(\frac{k_{-3}}{k_3}\right)^{1/\gamma} \cdot \sigma_2\right]$$

$$\Rightarrow \boxed{\frac{d\sigma_2}{d\tau} = \alpha\left[u_2 - \frac{k_{-3}}{k_2}\sigma_2^{\gamma}(1 - u_1 - u_2) + \frac{k_{-3}}{k_2}u_1\right] - \eta\sigma_2}$$

with $\boxed{\alpha = \frac{k_2 + k_{-1}}{k_1}\left(\frac{k_3}{k_{-3}}\right)^{1/\gamma}, \quad \eta = \frac{\gamma_2 (k_2 + k_{-1})}{e_0 k_1 k_2}}$

$$\frac{e_0 k_1 k_2}{(k_2 + k_{-1})^2}\frac{du_1}{d\tau} = -\sigma_1 u_1 + u_2 + \frac{k_{-3}}{k_2 + k_{-1}}\cdot\left[\sigma_2^{\gamma}(1 - u_1 - u_2) - u_1\right]$$

$$\frac{e_0 k_1 k_2}{(k_2 + k_{-1})^2}\frac{du_2}{d\tau} = \sigma_1 u_1 - u_2$$

$$\Rightarrow \boxed{\begin{cases} \varepsilon \frac{du_1}{d\tau} = -\sigma_1 u_1 + u_2 + \frac{k_{-3}}{k_2 + k_{-1}}\left[\sigma_2^{\gamma}(1 - u_1 - u_2) - u_1\right] \\ \varepsilon \frac{du_2}{d\tau} = \sigma_1 u_1 - u_2 \qquad \text{with } \varepsilon = \frac{e_0 k_1 k_2}{(k_2 + k_{-1})^2} \ll 1 \end{cases}}$$

(b) If $\varepsilon \ll 1$ we get the QSSA

$$\begin{cases} -\sigma_1 u_1 + u_2 + \frac{k_{-3}}{k_2 + k_{-1}}\left(\sigma_2^{\gamma}(1 - u_1 - u_2) - u_1\right) = 0 \\ \sigma_1 u_1 - u_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_2^{\gamma}(1 - u_1 - u_2) = u_1 \\ u_2 = \sigma_1 u_1 \end{cases} \Rightarrow \begin{cases} u_1 = \frac{\sigma_2^{\gamma}}{1 + \sigma_1\sigma_2^{\gamma} + \sigma_2^{\gamma}} \\ u_2 = \frac{\sigma_1\sigma_2^{\gamma}}{1 + \sigma_1\sigma_2^{\gamma} + \sigma_2^{\gamma}} \overset{\text{def}}{\equiv} f(\sigma_1, \sigma_2) \end{cases}$$

In terms of the remaining equations, the system becomes

$$\frac{d\sigma_1}{d\tau} = \nu - \frac{K_2+K_{-1}}{K_2}\cdot u_2 + \frac{K_{-1}}{K_2}u_2 = \nu - u_2 = \nu - f(\sigma_1,\sigma_2)$$

$$\frac{d\sigma_2}{d\tau} = \alpha\Big[\sigma_1 u_1 - \frac{K_{-3}}{K_2}\sigma_2^{\gamma}(1-u_1-\sigma_1 u_1) + \frac{K_{-3}}{K_2}u_1\Big] - \eta\sigma_2$$

$$= \alpha\Big[u_1\Big(\sigma_1 + \frac{K_{-3}}{K_2}\sigma_2^{\gamma} + \frac{K_{-3}}{K_2}\sigma_2^{\gamma}\sigma_1 + \frac{K_{-3}}{K_2}\Big) - \frac{K_{-3}}{K_2}\sigma_2^{\gamma}\Big] - \eta\sigma_2$$

$$= \alpha\Big[\frac{\sigma_2^{\gamma}}{1+\sigma_1\sigma_2^{\gamma}+\sigma_2^{\gamma}}\Big(\sigma_1 + \frac{K_{-3}}{K_2}(\sigma_2^{\gamma}\sigma_1+\sigma_2^{\gamma}+1)\Big) - \frac{K_{-3}}{K_2}\sigma_2^{\gamma}\Big] - \eta\sigma_2$$

$$= \alpha\Big[\underbrace{\frac{\sigma_1\sigma_2^{\gamma}}{1+\sigma_1\sigma_2^{\gamma}+\sigma_2^{\gamma}}}_{u_2} + \cancel{\frac{K_{-3}}{K_2}\sigma_2^{\gamma}} - \cancel{\frac{K_{-3}}{K_2}\sigma_2^{\gamma}}\Big] - \eta\sigma_2$$

$$= \alpha f(\sigma_1,\sigma_2) - \eta\sigma_2$$

Hence: $\begin{cases} \dfrac{d\sigma_1}{d\tau} = \nu - f(\sigma_1,\sigma_2) \\ \dfrac{d\sigma_2}{d\tau} = \alpha f(\sigma_1,\sigma_2) - \eta\sigma_2 \end{cases}$ is the QSSA

(c) Steady state : $\begin{cases} f(\sigma_1,\sigma_2) = \nu \\ \alpha f(\sigma_1,\sigma_2) = \eta\sigma_2 \end{cases} \Rightarrow \sigma_2 = \dfrac{\alpha\nu}{\eta}$

$$\frac{\sigma_1\sigma_2^{\gamma}}{1+\sigma_1\sigma_2^{\gamma}+\sigma_2^{\gamma}} = \nu$$

$$\sigma_1\sigma_2^{\gamma} = \nu(1+\sigma_1\sigma_2^{\gamma}+\sigma_2^{\gamma}) = \nu(\sigma_1\sigma_2^{\gamma}) + \nu(1+\sigma_2^{\gamma})$$

$$\Rightarrow \boxed{\sigma_1 = \frac{\nu(1+\sigma_2^{\gamma})}{\sigma_2^{\gamma}(1-\nu)} \quad \text{where } \sigma_2 = \frac{\alpha\nu}{\eta}}$$ is the unique steady state.

(d) Jacobian (J) $= \begin{bmatrix} -\dfrac{\partial f}{\partial\sigma_1} & -\dfrac{\partial f}{\partial\sigma_2} \\ \alpha\dfrac{\partial f}{\partial\sigma_1} & \alpha\dfrac{\partial f}{\partial\sigma_2} - \eta \end{bmatrix}$

Let $f_1 = \frac{\partial f}{\partial\sigma_1}$, $f_2 = \frac{\partial f}{\partial\sigma_2} \Rightarrow J = \begin{bmatrix} -f_1 & -f_2 \\ \alpha f_1 & \alpha f_2 - \eta \end{bmatrix}$

Char. polynomial: $\lambda^2 - (\alpha f_2 - \eta - f_1)\lambda + f_1\eta = 0$

Notice that $f_1 = \dfrac{\sigma_2^{\gamma} + \sigma_2^{2\gamma}}{(1+\sigma_1\sigma_2^{\gamma}+\sigma_2^{\gamma})^2} \geq 0$

Recall that if $\lambda^2 - a\lambda + b = 0$

then $\lambda_1 \cdot \lambda_2 = b$ for the roots λ_1, λ_2.
$\lambda_1 + \lambda_2 = a$

If $\lambda_1 \cdot \lambda_2 = f_1 \eta \geq 0 \Rightarrow$ both eigenvalues are of ~~the~~ same sign

Since ~~and~~ $\lambda_1 + \lambda_2 = \alpha f_2 - \eta - f_1$, we see that

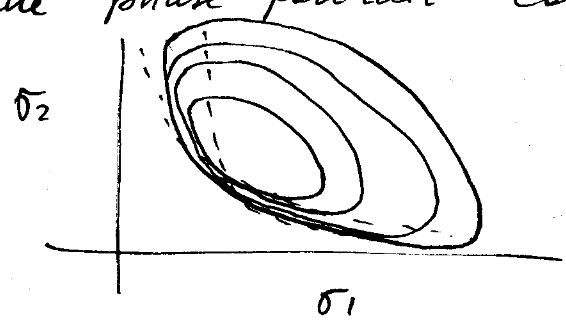
if $\alpha \frac{\partial f}{\partial \sigma_2} - \eta - \frac{\partial f}{\partial \sigma_1} < 0 \Rightarrow Re(\lambda_1 + \lambda_2) < 0 \Rightarrow Re(\lambda_1, \lambda_2) < 0$

if $\alpha \frac{\partial f}{\partial \sigma_2} - \eta - \frac{\partial f}{\partial \sigma_1} > 0 \Rightarrow Re(\lambda_1 + \lambda_2) > 0 \Rightarrow Re(\lambda_1, \lambda_2) > 0$.

Notice that even if the roots are complex, the condition $\alpha \frac{\partial f}{\partial \sigma_2} - \eta - \frac{\partial f}{\partial \sigma_1} < 0$ guarantees stability ($Re(\lambda_1, \lambda_2) < 0$) and otherwise the equilibrium is unstable.

It can be shown that $a = \alpha \frac{\partial f}{\partial \sigma_2} - \eta - \frac{\partial f}{\partial \sigma_1} = 0$ gives rise to Hopf bifurcations leading to periodic solutions.

(e) When $\nu = 0.0285$, $\eta = 0.1$, $\alpha = 1.0$, $\gamma = 2$
the phase portrait looks like a limit cycle:



Notice that this periodic orbit exists only in a small region of parameter space and will collapse if the values are perturbed.