Math 413 Fall 2010. Take-home Final Exam.
Due Monday Dec 20 by 5pm.

Collaboration on this exam is not allowed. All solution steps must be shown for full credit.

Part 1. Determine approximate solution to the following BVP using singular perturbation methods:

$$\begin{cases} \epsilon y'' + y' = 2x, & 0 < x < 1, 0 < \epsilon \ll 1 \\ y(0) = 1, y(1) = 1 \end{cases}$$

Part 2. Use Fourier transform to derive d'Alambert formula for the solution of the wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x) \end{cases}$$

Namely, show that $u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$

Hint: you may use the fact that $\cos(s) = (e^{is} + e^{-is})/2, \sin(s) = (e^{is} - e^{-is})/2i$ *and that* $\mathcal{F}(\psi) = ik\mathcal{F}(\chi)$, *where* $\chi(x) = \int_{-\infty}^{x} \psi(s)ds$.

Part 3. Consider the following model of glycolysis kinetics:

$$\begin{cases} \frac{ds_1}{dt} = \nu_1 - k_1 s_1 x_1 + k_{-1} x_2 \\ \frac{ds_2}{dt} = k_2 x_2 - k_3 s_2^{\gamma} e + k_{-3} x_1 - \gamma_2 s_2 \\ \frac{dx_1}{dt} = -k_1 s_1 x_1 + (k_{-1} + k_2) x_2 + k_3 s_2^{\gamma} e - k_{-3} x_1 \\ \frac{dx_2}{dt} = k_1 s_1 x_1 - (k_{-1} + k_2) x_2 \end{cases}$$

with the total amount of enzyme conserved: $x_1 + x_2 + e = e_0$.

(a) Nondimensionalize the problem using dimensionless variables $\sigma_1 = \frac{k_1 s_1}{k_2 + k_{-1}}, \sigma_2 = (\frac{k_3}{k_{-3}})^{1/\gamma} s_2, u_1 = x_1/e_0, u_2 = x_2/e_0, t = \frac{k_2 + k_{-1}}{e_0 k_1 k_2}\tau$.

(b) Show that if $\epsilon = e_0 k_1 k_2/(k_2 + k_{-1})^2 \ll 1$ one gets the following Quasi Steady State Approximation:

$$\begin{cases} \frac{d\sigma_1}{d\tau} = \nu - f(\sigma_1, \sigma_2) \\ \frac{d\sigma_2}{d\tau} = \alpha f(\sigma_1, \sigma_2) - \eta \sigma_2 \end{cases}$$

where $f(\sigma_1, \sigma_2) = \dfrac{\sigma_1 \sigma_2^\gamma}{\sigma_1 \sigma_2^\gamma + \sigma_2^\gamma + 1}$ and ν, α, η are constants related to the original parameters as $\nu = \dfrac{\nu_1}{e_0 k_2}, \alpha = \dfrac{k_2 + k_{-1}}{k_1} \left(\dfrac{k_3}{k_{-3}}\right)^{1/\gamma}, \eta = \dfrac{\gamma_2 (k_2 + k_{-1})}{e_0 k_1 k_2}$.

(c) Show that there is a unique steady state for the reduced system

(d) Use linear stability analysis to prove that there is a change of stability when

$$\alpha \frac{\partial f}{\partial \sigma_2} - \eta - \frac{\partial f}{\partial \sigma_1} = 0$$

(e) Sketch the phase portrait for $\nu = 0.0285, \eta = 0.1, \alpha = 1, \gamma = 2$. Comment on your observations.