Math 677.
Lecture 9.

Continuous dependence theorems:

Lemma. $\{f_n(t,x)\} \to f$ uniformly on any compact set in D, where f_n are continuous

Let $(\tau_n, \xi_n) \in D$
$$\downarrow$$
$$(\tau, \xi)$$

$\varphi_n(t)$ - noncontinuable sol. of $\begin{cases} \dot{x} = f_n(t,x) \\ x(\tau_n) = \xi_n \end{cases}$

If $\varphi(t)$ - unique sol. of $\begin{cases} \dot{x} = f(t,x) \\ x(\tau) = \xi \end{cases}$ on $[a,b]$

then $\varphi_n(t)$ - defined on $[a,b]$ for n - large and $\varphi_n \to \varphi$ uniformly on $[a,b]$.

$\boxed{\text{Thm.}}$ Let $f(t,x,\lambda)$ - continuous fct of (t,x,λ) for all $(t,x) \in D$, $D \in \mathbb{R}^n$ - open, $\forall \lambda$ close to λ_0

Let $\varphi(t,\tau,\xi,\lambda)$ be any noncontinuable solution of $\begin{cases} \dot{x} = f(t,x,\lambda) \\ x(\tau) = \xi \end{cases}$

If $\varphi(t,t_0,\xi_0,\lambda_0)$ is defined on $\underline{[a,b]}$ and is unique $\Rightarrow$ $\varphi(t,\tau,\xi,\lambda)$ defined on $[a,b]$ for all (τ,ξ,λ) near (t_0,ξ_0,λ_0) and is a continuous fct of (t,τ,ξ,λ).

Pf. By previous Lemma,

$|\varphi(t,\tau,\xi,\lambda) - \varphi(t_q,t_0,\xi_0,\lambda_0)| \le$

$|\varphi(t,\tau,\xi,\lambda) - \varphi(t,t_0,\xi_0,\lambda_0)| + |\varphi(t,t_0,\xi_0,\lambda_0) - \varphi(t_q,t_0,\xi_0,\lambda_0)|$

$\le 2\varepsilon$ for $|(t,\xi,\lambda) - (t_0,\xi_0,\lambda_0)| < \delta$

and $|t - t_q| < \delta_2$ $\forall a < t \le b$.

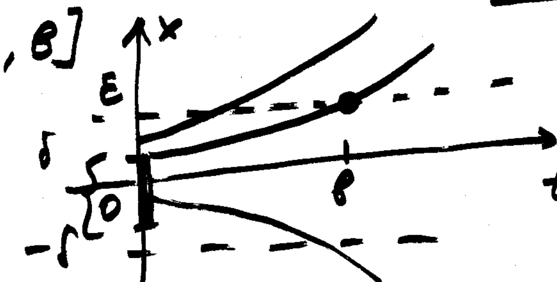
Ex. 1. $\begin{cases} \dot{x} = x \\ x(0) = x_0 \end{cases}$ $t_0 = 0$

$\varphi(t,t_0,x_0) = x_0 e^{t-t_0} = x_0 e^{t}$

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $x_0 < \delta \Rightarrow \varphi < \varepsilon$

$x_0 \cdot e^{t} < \varepsilon \Rightarrow x_0 < \varepsilon \cdot e^{-t} < \underline{\underline{\varepsilon e^{-b}}} = \delta$

Consider $t \in [0, b]$



Ex 2 $\begin{cases} \dot{x} = x^2 \\ x(0) = c \end{cases}$ (Demonstrates that continuity wrt initial data on inf. intervals is a question of stability).

$x(t,0,c) = -\dfrac{c}{ct-1}$ - solution to IVP

unique for $x(0) = c$

$x(t,0,0) = 0$ $\forall T > 0$ $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon, T) > 0$

s.t. $|x(t,0,c)| < \varepsilon$ if $|c| < \delta$ in $[0, T]$.

$0 < \delta < \delta_{max}$, δ_{max} satisfies $\dfrac{\delta}{1-\delta T} = \varepsilon \Rightarrow \delta = \dfrac{\varepsilon}{1+T\varepsilon}$

$\delta \to 0$ as $T \to \infty \Rightarrow$ no uniform continuity in t for $t \in (0, \infty)$.

Solution does not exist for $x(0) = c > 0$ on $[0, \infty)$

Want to develop stability theory for nonlinear systems.

General case: $\dot{x} = f(t,x)$ (*)

Special case: $\dot{x} = Ax \Rightarrow$ ~~φ_t~~ $\varphi_t = e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ defined a flow of a linear system.

$(x = e^{At} x_0)$

By Fundam. Thm.

~~Now~~: In linear case:

(i) $\varphi_0(x) = e^{A \cdot 0} \cdot x = x$

(ii) $\varphi_{s+t}(x) = e^{A(t+s)} \cdot x = \varphi_s(\varphi_t(x)) \quad \forall s,t \in \mathbb{R}$

(iii) $\varphi_{-t}(\varphi_t(x)) = \varphi_t(\varphi_{-t}(x)) \quad \forall t \in \mathbb{R}$

<u>Wanted</u>: flow definition for (*) with same properties.

<u>Def</u>. $f \in C^1(D)$, $D \subset \mathbb{R}^n$ - open

$x_0 \in D \quad \varphi(t, x_0)$ - solution to IVP $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$

defined on $I(x_0)$

$\Rightarrow \forall t \in I(x_0) \quad \varphi_t(x_0) = \varphi(t, x_0)$ - defines <u>a flow</u> of DE (*)

$\varphi(\cdot, x_0) : I \to D$ defines a trajectory of the system through x_0.

<u>Def</u>. $\Omega = \{(t, x_0) \in \mathbb{R} \times D \mid t \in I(x_0)\}$

<u>Lemma</u>. $f \in C^1(D)$, $D \subset \mathbb{R}^n$ - open

$\Rightarrow \Omega$ - open subset of $\mathbb{R} \times D$ and $\varphi \in C^1(\Omega)$.

Pf. If $(t_0, x_0) \in \Omega$, $t_0 > 0 \Rightarrow x(t) = \varphi(t, x_0)$ - sol. of IVP is defined on $[0, t_0]$.

$\Rightarrow$ sol. can be extended to $t_0 + \varepsilon$ for some $\varepsilon > 0$

$\Rightarrow \varphi(t, x_0)$ is defined on $[t_0 - \varepsilon, t_0 + \varepsilon] \Rightarrow$

$\exists N_\delta(x_0)$ s.t. $\varphi(t, y)$ is defined on $[t_0 - \varepsilon, t_0 + \varepsilon] \times N_\delta(x_0)$

$\Rightarrow \underbrace{(t_0 - \varepsilon, t_0 + \varepsilon) \times N_\delta(x_0)} \subset \Omega \Rightarrow \Omega$ - open in $\mathbb{R} \times D$.

let $G =$ (neighborhood of (t_0, x_0))

$f \in C'(G)$ for all t_0 $\Rightarrow$ in a similar way t_0 can be picked < 0, so $f \in C'(\Omega)$. (t_0, x_0) - arbitrary.

Corollary:

$f \in C^k(D) \Rightarrow \varphi \in C^k(D)$.

f - analytic on $D \Rightarrow \varphi$ - analytic on D.

Thm. $E \subset \mathbb{R}^n$ - open $f \in C'(E)$

$\Rightarrow \forall x_0 \in E$ if $t \in I(x_0)$ and $s \in I(\varphi_t(x_0))$

then $\varphi_{t+s}(x_0) = \varphi_s(\varphi_t(x_0))$, $s + t \in I(x_0)$.

Pf. Suppose $s > 0$ $t \in I(x_0)$, $s \in I(\varphi_t(x_0))$, $\varphi_t(x_0) = \varphi(t, x_0)$

$I(x_0) = (\alpha, \beta)$

Define $x : (\alpha, s+t] \to E$ by

$$x(r) = \begin{cases} \varphi(r, x_0), & \alpha < r \le t \\ \varphi(r - t, \varphi_t(x_0)) & t \le r \le s + t \end{cases}$$

$x(r)$ - sol. to IVP on $(\alpha, s+t]$. $\Rightarrow s + t \in I(x_0)$

By uniqueness $\varphi_{s+t}(x_0) = x(s+t) = \varphi(s, \varphi_t(x_0))$
$= \varphi_s(\varphi_t(x_0))$

$s < 0$, $s = 0$ - trivial.

Remark 1. By time rescaling, we can show that (i)-(ii) hold for any $t \in \mathbb{R}$.

Def. $S \subset \mathbb{R}^n$ - open is called invariant wrt flow φ if $\varphi(S) \subset S$ $\forall t \in \mathbb{R}$.

(ex: stable/unstable/center manifolds) introduced for linear systems earlier.