Math 677.
Lecture 8.

HW problem:

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \qquad Av = \lambda v, \quad v = \begin{pmatrix} -b \\ a \end{pmatrix}$$

Show: if x_0 belongs to a line $kx^{(1)} + p = x^{(2)}$ and $x(t)$ belongs to the same line (in other words, this is an invariant subset in the solution) then $k = -\frac{a}{b}$, $p = 0$.

We know: $x(t) = e^{At}x(0)$ $\qquad$ v_1, v_2 - (gen.) eig. vectors; λ_1, λ_2 - eig. values

let $v_2 = v = \begin{pmatrix} -b \\ a \end{pmatrix}$ eig. v.

$$\Rightarrow \{v_1, v_2\} - \text{Basis in } \mathbb{R}^2$$

$$x(0) = c_1 v_1 + c_2 v_2 = c_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} + c_2 \begin{pmatrix} -b \\ a \end{pmatrix}$$

$$x(t) = e^{At}x(0) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2 =$$

$$c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 = c_1 e^{\lambda_1 t} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} -b \\ a \end{pmatrix}$$

If $k \cdot x_0^{(1)} + p = x_0^{(2)} \Rightarrow$

$$e^{\lambda t} \times \left[k \cdot (c_1 v_{11} - b c_2) + p = c_1 v_{12} + c_2 a \right]$$

If $k \cdot x^{(1)}(t) + p = x^{(2)}(t) \Rightarrow$

$$-\begin{cases} k(\cancel{c_1 e^{\lambda_1 t}} v_{11} - b c_2 e^{\lambda_2 t}) + p = \cancel{c_1 v_{12} e^{\lambda_1 t}} + c_2 a e^{\lambda_2 t} \\ k(\cancel{c_1 e^{\lambda_1 t} v_{11}} - b c_2 e^{\lambda_1 t}) + p e^{\lambda_1 t} = \cancel{c_1 v_{12} e^{\lambda_1 t}} + c_2 a e^{\lambda_1 t} \end{cases}$$

$$\Rightarrow k b c_2 (\underline{e^{\lambda_1 t}} - e^{\lambda_2 t}) + p(\underline{e^{\lambda_1 t}} - 1) = a c_2 (e^{\lambda_2 t} - \underline{e^{\lambda_1 t}})$$

$$\begin{cases} kbc_2 + p = -ac_2 \\ -kb\cancel{c_2} = a\cancel{c_2} \end{cases} \quad \begin{bmatrix} p = 0 \\ k = -\frac{a}{b} \end{bmatrix} \quad \blacksquare$$

We know: $f \in C^1 \Rightarrow \exists!$ sol. to IVP (locally)

f - globally Lipschitz $\Rightarrow$ sol. depends continuously on initial data.

Continuation of solutions.

Thm. $f \in C(D)$, $|f| \le M$ on D

φ - sol. of $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ on $J = (a, b)$

Then

(i) $\exists \lim\limits_{t \to a+} \varphi(t) = \varphi(a^+)$, $\exists \lim\limits_{t \to b-} \varphi(t) = \varphi(b-)$

(ii) if $(a, \varphi(a+)) \in D \Rightarrow \varphi$ can be continued to the left passing through $t = a$

if $(b, \varphi(b-)) \in D \Rightarrow \varphi$ can be continued to the right of $t = b$.

Pf. 1) Consider b

Fix $\tau \in J$, $\xi = \varphi(\tau)$

$\tau < u < t < b \Rightarrow |\varphi(u) - \varphi(t)| = \left|\int_t^u f(s, \varphi(s))ds\right| \le$
$\le M|u - t|$

since $\dot{\varphi} = f(t, \varphi)$ means that

$$\varphi(t) - \varphi(u) = \int_u^t f(s, \varphi(s))\, ds$$

It means that $\{t_m\} \uparrow b \Rightarrow \varphi(t_m)$ - Cauchy sequence

$\Rightarrow \exists \lim\limits_{t \to b-} \varphi(t) = \varphi(b-)$

2) Suppose $(b, \varphi(b-)) \in D$.

By local existence Thm, we can extend solution φ to $b+\delta$ for some $\delta > 0$ by doing the following:

take $\tilde{\varphi}(t)$ solution of IVP $\begin{cases} \dot{x} = f(t,x) \\ x(b) = \varphi(b-) \end{cases}$

$b \le t \le b+\delta$

take $\varphi^*(t) = \begin{cases} \varphi(t), & a \le t \le b \\ \tilde{\varphi}(t), & b \le t \le b+\delta \end{cases}$

φ^* is the solution to the original IVP. ∎

<u>Corollary 1</u>.

Notation: $\boxed{(t, \varphi(t)) \to \partial D \text{ as } t \to \partial J^*}$ if for any compact set $K \subseteq D$ $\exists t^* \in J$ s.t. $(t, \varphi(t)) \notin K$ for all $t \in (t^*, \partial J^*)$.

J^* - maximal interval of existence.

<u>Claim</u>: If $f \in C(D)$, φ - sol. to IVP on J

$\Rightarrow$ f - can be continued to the maximal interval $J^* \supset J$ in such a way that $(t, \varphi(t)) \to \partial D$ as $t \to \partial J^*$ and $|t| + |\varphi(t)| \to \infty$ if ∂D is empty. The extended solution φ^* on J^* is noncontinuable.

Sketch of the proof:

Step 1. By Zorn's lemma, J^* exists.

By Thm 1, $J^* = (a, b)$ - must be open.

If $b = \infty$ then $(t, \varphi^*) \to \partial D$ as $t \to \infty$.

Step 2. Take $b < \infty$.

Let $a < c < b$.

Can show: $\forall K \subseteq D$ compact $\{(t, \varphi^*) : t \in [c, b)\}$ is not contained in K.

Suppose otherwise $\Rightarrow f(t, \varphi^*)$ - bdd in $[c, b)$ and $(b, \varphi^*(b-)) \in K \subseteq D$

By Thm 1 we can continue beyond $b \Rightarrow$ contradiction.

Step 3. Show: $(t, \varphi^*) \to \partial D$ as $t \to b-$.

skip details.

(use step 2 & Thm 1 to continue until ∂D is reached).

Corollary 2. If solution $\varphi(t)$ of IVP is bounded $|\varphi(t)| \leq M$ whenever it exists $\Rightarrow$ then it exists for all t.

Reason: $D_1 = \{(t, x) : t_0 \leq t \leq T, |x| \leq M\}$ for any $T > t_0$.

Then $f(t, x)$ bdd in $D_1 \Rightarrow$

$x(t)$ can be continued to ∂D_1

$x(t)$ exists for all $t_0 \leq t \leq T$, T-arbitrary

Same way for $-T \leq t \leq t_0 \Rightarrow \boxed{J^* = (-\infty, \infty)}$

Corollary 3. $\begin{cases} \dot{x} = A(t)x + g(t) = f(t,x) \\ x(t_0) = x_0 \end{cases}$

If $A(t), g(t)$ - continuous at $t \in I \Rightarrow \exists !$ solution in I.

Reason: $\varphi(t)$ - sol. for $t_0 \le t \le t_0 + c \Rightarrow$

$$\varphi(t) - x_0 = \int_{t_0}^{t} f(s, \varphi(s))\, ds = \int_{t_0}^{t} \big(f(s,\varphi(s)) - f(s,x_0) + f(s,x_0)\big) ds$$

$$\Rightarrow \quad |\varphi(t) - x_0| \le \int_{t_0}^{t} \|A(s)\| \cdot |\varphi(s) - x_0|\, ds + \int_{t_0}^{t} |f(s,x_0)|\, ds$$

since $f(s,\varphi(s)) - f(s,x_0) = A(s)\cdot\varphi(s) + \cancel{g(t)} - A(s)\cdot x_0 - \cancel{g(t)}$

$$\le \delta + L \int_{t_0}^{t} |\varphi(s) - x_0|\, ds$$

$L = \sup\limits_{t_0 \le s \le t_0 + c} \|A(s)\|$, $\quad \delta = \max\limits_{t_0 \le s \le t_0 + c} |A(s)x_0 + g(s)| \cdot c$

By Gromwall lemma: $|\varphi(t) - x_0| \le \delta e^{Lc}$

$\Rightarrow \varphi$ - bdd on $[t_0, t_0 + c) \Rightarrow$ can be extended beyond $t = c$.