Math 677.
Lecture 7.

Lemma (Gronwall inequality)

$\lambda(t) \in C(\mathbb{R})$ $\quad \mu(t) > 0$ continuous

If $y(t)$ is s.t. $y(t) \le \lambda(t) + \int_a^t \mu(s) y(s)\,ds$ $\quad a \le t \le b$

then for any $t \in [a, b]$

$$y(t) \le \lambda(t) + \int_a^t \mu(s)\lambda(s) e^{\int_s^t \mu(\tau)d\tau} ds$$

In particular, if $\lambda(t) \equiv \lambda$ then

$$y(t) \le \lambda \exp\left(\int_a^t \mu(s)\,ds\right)$$

Proof: ① $z(t) = \int_a^t \mu(s) y(s)\,ds$

$$z'(t) = \mu(t) y(t) \le \mu(t) \cdot (\lambda(t) + z(t))$$

$$\Rightarrow z'(t) - \mu(t) z(t) \le \lambda(t)\mu(t)$$

② Let $w(t) = z(t) e^{-\int_a^t \mu(\tau)d\tau}$

$$w'(t) = z'(t) e^{-\int_a^t \mu(\tau)d\tau} - z\mu(t)\cdot e^{-\int_a^t \mu(\tau)d\tau}$$

$$= e^{-\int_a^t \mu(\tau)d\tau}(z' - z\mu) \underset{\text{by ①}}{\le} \lambda(t)\mu(t) e^{-\int_a^t \mu(\tau)d\tau}$$

$$w(a) = z(a) \cdot e^0 = 0$$

③ Integrate ② to get

$$\underbrace{w(t)}_{= z \cdot \exp(-\int_a^t \mu(\tau)d\tau)} \le \int_a^t \lambda(s)\mu(s) \exp\left(-\int_a^s \mu(\tau)d\tau\right) ds$$

$$z(t) \le \int_a^t \lambda(s)\mu(s) \exp\left(\int_s^t \mu(\tau)d\tau\right) ds$$

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left(\int_s^t \mu(\tau)d\tau\right) ds \quad \blacksquare$$

Thm 1 (Uniqueness of solution to IVP)

x_1, x_2 - differentiable s.t. $|x_1(a) - x_2(a)| \leq \delta$

and $|x_i'(t) - f(t, x_i(t))| \leq \varepsilon_i$, $i = 1, 2$ for $a \leq t \leq b$

If f is Lipschitz with constant L on $[a, b]$, then

$$|x_1(t) - x_2(t)| \leq \delta e^{L(t-a)} + (\varepsilon_1 + \varepsilon_2)(e^{L(t-a)} - 1)/L$$

Proof: $\varepsilon = \varepsilon_1 + \varepsilon_2$

Take $\gamma(t) = x_1(t) - x_2(t) \Rightarrow |\gamma(a)| = |x_1(a) - x_2(a)| \leq \delta$

① $|\gamma'(t)| = |x_1'(t) - x_2'(t)| =$

$$= |x_1'(t) - f(t, x_1) + f(t, x_1) - x_2'(t) + f(t, x_2) - f(t, x_2)|$$

$$\leq |x_1'(t) - f(t, x_1)| + |x_2'(t) - f(t, x_2)| + |f(t, x_1) - f(t, x_2)| \leq \varepsilon + L|x_1 - x_2|$$

$$= \varepsilon + L|\gamma(t)|$$

$\cancel{\int_a^t |\gamma'(s)| ds}$

② $\int_a^t \gamma'(s)\, ds = \gamma(t) - \gamma(a)$

$$\Rightarrow |\gamma(t)| = \left|\int_a^t \gamma'(s)\, ds + \gamma(a)\right| \leq |\gamma(a)| + \left|\int_a^t \gamma'(s)\, ds\right|$$

$$\leq |\gamma(a)| + \int_a^t |\gamma'(s)|\, ds \leq$$

$$\underset{\text{by ①}}{\leq} |\gamma(a)| + \int_a^t (\varepsilon + L|\gamma(s)|)\, ds =$$

$$= \underbrace{|\gamma(a)|}_{\leq \delta} + \varepsilon(t-a) + \int_a^t L|\gamma(s)|\, ds$$

③ By Gronwall lemma,

$$|\gamma(t)| \leq \delta + \varepsilon(t-a) + \int_a^t L(\delta + \varepsilon(s-a)) e^{\int_s^t L\, d\tau} ds$$

$$\Rightarrow \int_a^t L(\delta + \varepsilon(s-a)) \cdot e^{L(t-s)}\, ds =$$

$$= (\delta + \varepsilon(t-a))(-e^{L(t-t)}) + e^{L(t-a)} \cdot \delta + \frac{\varepsilon}{L}(e^{L(t-a)} - 1)$$

$$\Rightarrow |y(t)| \le \cancel{\delta} + \cancel{\varepsilon(t-a)} \cancel{- \delta} - \cancel{\varepsilon(t-a)} + + \delta e^{L(t-a)} + \frac{\varepsilon}{L}(e^{L(t-a)} - 1) \quad \blacksquare$$

Corollary:

If x_1, x_2 – solutions to IVP $\dot{x} = f(t,x)$, then $\varepsilon_1 = \varepsilon_2 = \delta = 0$ so $|x_1 - x_2| = 0 \Rightarrow$ uniqueness of solutions to IVP for $f \in Lip(L)$ on $[a,b]$.

Corollary 2: $f \in C^1(D) \Rightarrow$ same statements hold. Namely, $\exists!$ sol. to IVP

Suffices to show that $f \in Lip(L)$ in D.

Suppose $f \in C^1(D)$, $(t_0, x_0) \in D$ then $D_x f(t,x) \in C$ in $R = \{|t - t_0| \le \delta_1, |x - x_0| \le \delta_2\}$

$\uparrow$ rectangular region

$$f(t,x_1) - f(t,x_2) = \int_0^1 \frac{d}{ds} f(t, s x_1 + (1-s)x_2)\, ds$$

$$= \int_0^1 \underbrace{D_x f(t, s x_1 + (1-s)x_2)}_{\le M \text{ on } R,\ 0 \le s \le 1} \cdot (x_1 - x_2)\, ds$$

$$M = \sup_{(t,x)} D_x f(t, s x_1 + (1-s)x_2)$$

$$\Rightarrow |f(t,x_1) - f(t,x_2)| \le M|x_1 - x_2| \Rightarrow f \in Lip(M) \text{ on } D.$$

Corollary 3. (Continuous dependence on initial data)

For any interval $[t_0, t_0+T]$ any a Lipschitz f, $f \in Lip(L)$

Given $\varepsilon > 0$ $\exists \delta = \delta(\varepsilon, T) > 0$ s.t. $\dot{x}_i = f(t, x_i)$, $i=1,2$

$$|x_1(t_0) - x_2(t_0)| < \delta \Rightarrow |x_1(t) - x_2(t)| < \varepsilon$$

for all $t \in [t_0, t_0+T]$

Proof: take $\varepsilon_1 = \varepsilon_2 = 0$ (in Thm 1)

$$\Rightarrow |x_1(t) - x_2(t)| \le \delta e^{L(t-t_0)} \quad \text{for } t \ge t_0$$

Choose $\delta < \varepsilon e^{-LT} \Rightarrow |x_1(t) - x_2(t)| < \varepsilon$

Jordan forms.

$$f(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)^{n_1} \dots (\lambda - \lambda_s)^{n_s}$$

n_i - algebraic multiplicity of λ_i

$V_i = \{x \in \mathbb{R}^n \mid Ax = \lambda_i x\}$ — eigenspace of λ_i.

$V_i = Null(A - \lambda_i I) = N(A - \lambda_i I)$

m_i - geometric multiplicity of $\lambda_i = \dim V_i$

$\boxed{m_i \le n_i}$ defect$(\lambda_i) = \underbrace{n_i - m_i}_{\text{\# of gen. eigenvectors to be computed}}$

$M(\lambda_i, A) = N(A - \lambda_i I)^{n_i}$ - generalized eigenspace

$$\mathbb{R}^n = M(\lambda_1, A) \oplus \dots \oplus M(\lambda_s, A)$$

$$N(A - \lambda_i I) \subseteq N(A - \lambda_i I)^2 \subseteq N(A - \lambda_i I)^3 \subseteq \dots$$

$$N(A - \lambda_i I)^{n_i} = N(A - \lambda_i I)^{n_i + 1}$$