Math 677.
Lecture 6.

$\dot{x} = f(x)$ – nonlinear ODEs $(\vec{x} = (x_1, \dots x_n, t))$

if f does not dep. on $t \Rightarrow$ autonomous system

In nonlinear case:

1) if f is continuous $\Rightarrow$ exists a solution but can be non-unique

ex: $\begin{cases} \dot{x} = 3x^{2/3} \\ x(0) = 0 \end{cases}$

$x_1 \equiv 0$
$x_2 = t^3$ through $(0,0)$

2) solutions can be unbounded

$\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$ $\quad x(t) = \dfrac{1}{1-t} \quad t \neq 1 \quad t_0 = 0$

maximal interval of existence: $(-\infty, 1)$

$t=1$

Let us define some relevant quantities:

1) $Df(x_0)(x) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x_0) \cdot x_j \leftarrow$ derivative of f at x_0

suppose f – differentiable at x_0 i.e.

$$\lim_{|h| \to 0} \frac{|f(x_0 + h) - f(x_0) - Df(x_0)h|}{|h|} = 0$$

$Df(x_0) = \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^{n}$ – Jacobian of f.

2) $f \in C^1(\mathbb{R})$ - continuously differentiable

3) $F : V_1 \to V_2$ - continuous map if
$\quad \| \ \|_1 \quad \| \ \|_2$ (at x)

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. ~~$\forall x \in V_1$~~, $y \in V_1$ s.t.

$$\|x - y\|_1 < \delta \Rightarrow \|F(x) - F(y)\|_2 < \varepsilon$$

4) $f \in C^1(E) \Leftrightarrow \frac{\partial f_i}{\partial x_j}$ - cts in E

$f \in C^k(E) \Leftrightarrow \dfrac{\partial^k f_i}{\partial x_{j_1} \dots \partial x_{j_k}}$ - cts in E

$$D^2 f(x_0)(x, y) = \sum_{j_1, j_2 = 1}^{n} \frac{\partial^2 f}{\partial x_{j_1} \partial y_{j_2}} x_{j_1} y_{j_2}$$

5) $f : E \to \mathbb{R}^n$ analytic in $E \subset \mathbb{R}^n$
$(f_1, \dots f_n)$ if f_j - analytic in E for $j = 1, \dots n$
Taylor series conv. to f_j ~~in~~.

6) $A : M \to M$
metric space with metric ρ
if $\exists \lambda \in (0,1)$ s.t. $\rho(Ax, Ay) \leq \lambda \rho(x,y)$ $\forall x, y \in M$
then A - contraction.

__example__ : ① $A : \mathbb{R}^n \to \mathbb{R}^n$ $|\lambda| < 1$
matrix gives contraction

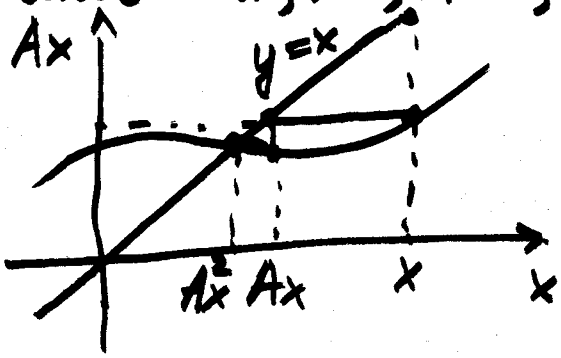
② $A : \mathbb{R} \to \mathbb{R}$
if $|A'(x_0)| < 1$ A is not necessary a contraction
But if $|A'(x_0)| \leq \delta < 1 \Rightarrow A$ - contraction

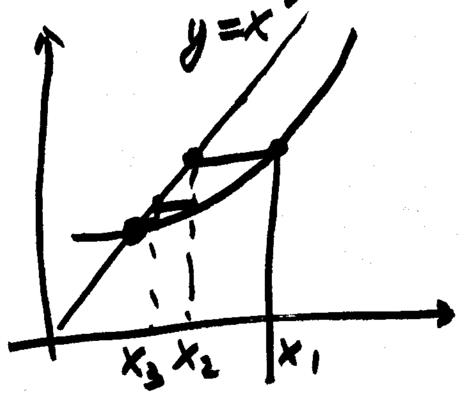
Thm. (Brauer)

$A: M \to M$ ~~continuous~~ contraction

$\Rightarrow \exists!$ fixed point x (s.t. $Ax^* = x^*$)

and $x, Ax, A^2x, \ldots$ converges to x^*.





Proof $d = \rho(x, Ax)$

$\rho(A^n x, A^{n+1} x) \le \lambda^n d \qquad 0 < \lambda < 1$

$\sum_{n=0}^{\infty} \lambda^n < \infty \Rightarrow A^n x$ - Cauchy sequence

M - complete

$\Rightarrow A^n x \underset{n \to \infty}{\longrightarrow} x^*$ (convergence is shown)

To show: x^* is a fixed pt

$$Ax^* = A \lim_{n \to \infty} A^n x = \lim_{n \to \infty} A^{n+1} x = x^*$$

If y is another fixed point, i.e. $Ay = y$

then $y = x^*$

$\rho(x^*, y) = \rho(Ax^*, Ay) \le \lambda \rho(x^*, y) \quad \lambda < 1$

$\Rightarrow \rho(x^*, y) = 0 \Rightarrow x^*$ is a unique fixed point.

$\{A^k x\}_{k=0}^{\infty}$ - successive approximation to x^*.

$\forall y \in \{A^k x\}, \rho(y, x^*) \le \frac{d}{1-\lambda}, \quad d + \lambda d + \ldots = \frac{d}{1-\lambda}$

<u>Thm.</u> (Fundam. Thm of Existence & Uniqueness of Solutions to IVP.)

Local result

Let $f \in C^1(E)$, $E \subset \mathbb{R}^n$ - open subset
$x_0 \in E$

$\Rightarrow \exists a > 0$ s.t. solution to IVP $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$

exists and is unique on $[-a, a]$.

<u>Sketch of the proof</u>:

Consider $\underset{\substack{\wr\wr \\ x(t)-x_0}}{(Ah)(t,x)} = \int_{t_0}^{t} f(\tau, x + h(\tau, x))\, d\tau$

1) To show: A - contraction mapping in $[-a, a]$

$\|Ah_1 - Ah_2\|(t,x) \leq \lambda \cdot \|h_1 - h_2\|$ ← we want to show
for some $0 < \lambda < 1$

$$|Ah_1 - Ah_2| = \left|\int_{t_0}^{t} \left(f(\tau, x+h_1) - f(\tau, x+h_2)\right) d\tau\right| \leq$$

$$\leq \int_{t_0}^{t} |\underbrace{f(\tau, x+h_1) - f(\tau, x+h_2)}_{\leq\, L|h_1 - h_2|}|\, d\tau \leq L \int_{t_0}^{t} |h_1 - h_2|\, d\tau$$

since $f \in C^1$
f is Lipschitz

$\leq La\|h_1 - h_2\|$

Pick $\lambda = La < 1$ (i.e. $a < \frac{1}{L}$) $\Rightarrow$ contraction.

2) $\Rightarrow A$ has a fixed pt h s.t. $Ah = h$

Take $g(t,x) = x + h(t,x) =$

$$= x + \int_{t_0}^{t} f(\tau, x+h)\, d\tau \Rightarrow \begin{cases} \frac{\partial g}{\partial t} = f(t, g(t)) \\ g(0) = x_0 \end{cases} \Rightarrow \text{sol. to IVP.}$$

<u>Existence is shown.</u>