Math 677.
Lecture 5

Stability theory for linear systems.

$\dot{x} = Ax$

$w_j = u_j + i v_j$ – gen. eigenvectors of A corresp. to $\lambda_j = a_j + i b_j$

$B = (u_1 \ldots u_k\, u_{k+1}\, v_{k+1} \ldots u_n\, v_n)$ – basis for $\mathbb{R}^n$

<u>**Def.**</u> of eigenspaces E^s, E^u, E^c:

$E^s = \{x \mid x \in \mathrm{Span}\{u_j, v_j \mid a_j < 0\}\}$

$= \mathrm{Span}\{u_j, v_j \mid a_j < 0\}$ – stable manifold (subspace)

$E^u = \mathrm{Span}\{u_j, v_j \mid a_j > 0\}$ – unstable

$E^c = \mathrm{Span}\{u_j, v_j \mid a_j = 0\}$ – center manifold

(comes for $e^{a_j t}$ terms in solution)

$t^\alpha e^{a_j t} \cos b_j t$ or $t^\alpha e^{a_j t} \sin b_j t$

<u>Ex.</u> $A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad w_1 = u_1 + i v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

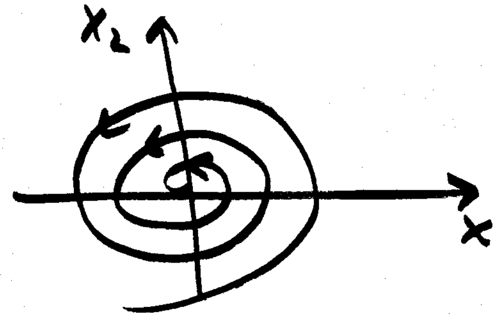
$\lambda_1 = -2 + i$

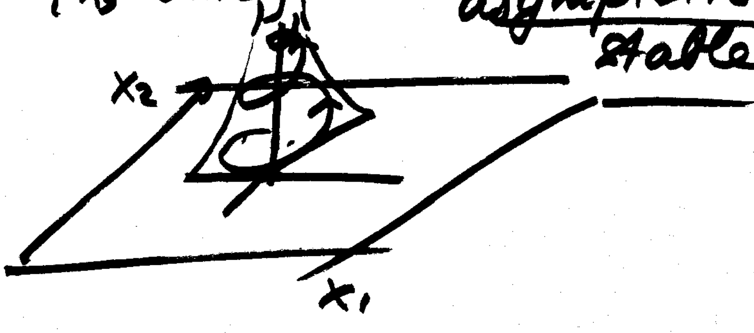
$w_2 = u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \lambda_2 = 3$

$E^s = \mathrm{span}\{u_1, v_1\} = \{x_1 x_2\text{-plane}\}$

$E^u = \mathrm{span}\{u_2\} = \{x_3\text{-axis}\}$

asymptotically stable



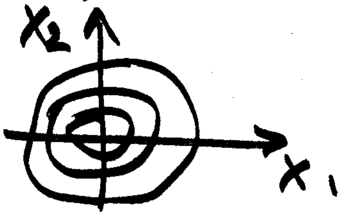


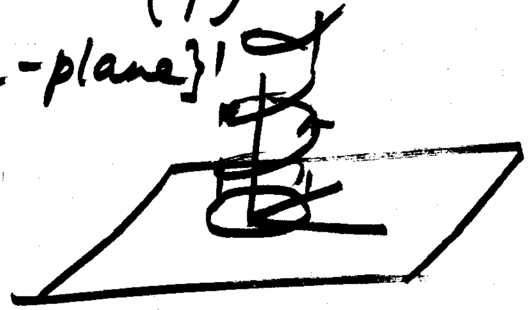
Ex. $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad \lambda_1 = i \quad u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = 2 \quad u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$E^c = \text{span}\{u_1, v_1\} = \{x_1 x_2\text{-plane}\}$

$E^u = \{x_3\text{-axis}\}$



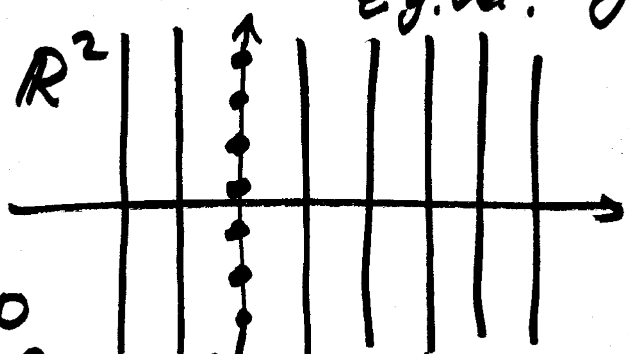


stable

Ex. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \lambda_1 = \lambda_2 = 0 \quad u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

eig.vec. gen. eig.vec.

$E^c = \text{span}(u_1, u_2) = \mathbb{R}^2$



$x_1 = c_1$

$x_2 = c_1 t + c_2$

if $c_1 = 0 \Rightarrow x_1 = 0$

$x_2 = c_2$ ← bounded

if $c_1 \neq 0 \Rightarrow x_2 \to \infty$ as $t \to \infty$ unbounded

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Def. If $\lambda_1, \dots \lambda_n$ are eigenvalues of A s.t. $Re(\lambda_i) \neq 0$ for all $i = 1, \dots n$

$\Rightarrow e^{At}$ - hyperbolic flow

and $\dot{x} = Ax$ is called hyperbolic system

Recall: $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$

$x_0 \to x(t) = e^{At} x_0$ called a flow of ODE

Lemma. E – gen. eigenspace corresp. to λ then E is A-invariant (AECE)

Proof:

$\{v_1, \dots v_k\}$ – basis of E

$\forall v \in E \quad v = \sum_{j=1}^{k} c_j v_j$

$Av = \sum_{j=1}^{k} c_j \underline{Av_j}$ Show: $Av_j \in E$:

$(A - \lambda I)^{k_j} v_j = 0$ for some k_j

$(A - \lambda I) v_j = V_j \Rightarrow V_j \in Ker (A - \lambda I)^{k_j - 1} \subset E$

Then $Av_j = \lambda v_j + V_j \in E \Rightarrow Av \in E$

Thm. $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$ where E^s, E^u, E^c are e^{At}-invariant

Proof. Take basis $(u_1, \dots u_m, u_{m+1}, v_{m+1}, \dots u_n, v_n)$ of $\mathbb{R}^n$

Let $x_0 \in E^s \Rightarrow e^{At} x_0 \in E^s$

Suppose $x_0 \in E^s$ looks like $x_0 = \sum_{j=1}^{n_s} c_j v_j$

$E^s = span\{v_1, \dots v_{n_s}\}$

$$\Rightarrow e^{At} x_0 = \lim_{k \to \infty} \left(I + At + \dots + \frac{A^k t^k}{k!}\right) \cdot \sum_{j=1}^{n_s} c_j v_j$$

$$= \sum_{j=1}^{n_s} c_j \lim_{k \to \infty} \left(I + At + \dots + \frac{A^k t^k}{k!}\right) v_j \in E_s$$

Because each $A^k v_j \in E_s$

Theorem 1 TFAE

(a) $\forall x_0 \in \mathbb{R}^n \ \lim_{t\to\infty} e^{At}x_0 = 0 \quad \forall x_0 \neq 0 \ \lim_{t\to-\infty} |e^{At}x_0| = \infty$

(b) $\mathrm{Re}|\lambda_j| < 0$

(c) $\exists a>0, c>0, m>0, M>0$ s.t. $\forall x_0 \in \mathbb{R}^n$

$$|e^{At}x_0| \geq m e^{-at}|x_0| \quad \forall t \leq 0$$
$$|e^{At}x_0| \leq M e^{-ct}|x_0| \quad \forall t \geq 0$$

Theorem 2, TFAE

(a) $\forall x_0 \in \mathbb{R}^n \ \lim_{t\to-\infty} e^{At}x_0 = 0, \ \forall x_0 \neq 0 \ \lim_{t\to+\infty} |e^{At}x_0| = \infty$

(b) $\mathrm{Re}|\lambda_j| > 0$

(c) $\exists a, c, m, M > 0$ s.t. $\forall x_0 \in \mathbb{R}^n$

$$|e^{At}x_0| \leq M e^{ct}|x_0| \quad t \leq 0$$
$$|e^{At}x_0| \geq m e^{at}|x_0| \quad t \geq 0$$

Nonhomogeneous Linear Systems

$$\dot{x} = Ax + B(t) \ (1), \qquad \dot{x} = Ax \ (2)$$

Def. Fundamental matrix of solutions of (2) $\Phi(t)$ is any nonsingular matrix function s.t. $\Phi'(t) = A\Phi(t) \ \forall t \in \mathbb{R}$

(Ex. $\Phi(t) = e^{At}$)

$\Rightarrow$ All fundam. matrices are given by $\Phi(t) = e^{At}C$ where $|C| \neq 0$.

⇒ <u>Theorem 3</u> $\Phi(t)$ – fundam. matrix for $\dot{x} = Ax$

then solution of (1) for $x(0)=x_0$ is unique and is given by

$$x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)b(\tau)\,d\tau$$

Special case: $\Phi(t) = e^{At}$ ⇒

$$\boxed{x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}b(\tau)\,d\tau}$$

Remark: $A = A(t)$ gives the same result.