Math 677.
Lecture 4

Jordan forms.

$A \in \mathbb{R}^{n\times n}$ $\quad \lambda_1, \dots \lambda_k, \underbrace{\lambda_{k+1} \dots \lambda_n}$

real $\quad$ complex $\lambda_j = a_j \pm i b_j$

$v_1, \dots v_k, w_1, \dots w_n,$

generalized eigenvectors $\quad w_j = u_j \pm i v_j$

$\Rightarrow \quad P = [v_1 \dots v_k \, v_{k+1} \, u_{k+1} \dots v_n \, u_n]$

is invertible and

$$J = P^{-1} A P = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_r \end{bmatrix}$$

Block-diagonal form

J – Jordan canonical form

B_j – Jordan blocks.

B_j can be either of the form

(a) $B = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix}$ OR

(b) $B = \begin{pmatrix} D & I_2 & & 0 \\ & \ddots & \ddots & \\ & & & I_2 \\ 0 & & & D \end{pmatrix}$

$\lambda = a + ib$

$D = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

I_2 – identity of size 2×2

If $\dot{x} = Ax$, $x(0) = x_0$

Then $x(t) = e^{At} x_0$ from fund. theorem

$$\Rightarrow x(t) = e^{PJP^{-1}t}x_0 = Pe^{Jt}P^{-1}x_0 =$$

$$= P\begin{bmatrix} e^{B_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{B_r t}\end{bmatrix}P^{-1}x_0$$

How to compute e^{B_j} ?

<u>Case (a)</u>: $\underset{\substack{m\times m \\ \text{matrix}}}{B} = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} = \lambda I + N$, where $N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

$$e^{Bt} = e^{\lambda t} \cdot e^{Nt}$$

$$e^{Nt} = I + Nt + \frac{N^2}{2}t^2 + \dots + \frac{N^{m-1}t^{m-1}}{(m-1)!}$$

since $N^2 = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \\ & & & & 0 \end{pmatrix} \dots \quad N^m = \theta$

$$\Rightarrow e^{Nt} = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{m-1}}{(m-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2} \\ & \Theta & & \ddots & t \\ & & & & 1 \end{bmatrix}$$

$$e^{Bt} = e^{\lambda t}\begin{bmatrix} 1 & t & \dots & \frac{t^{m-1}}{(m-1)!} \\ & \ddots & \ddots & \\ & 0 & \ddots & t \\ & & & 1 \end{bmatrix}$$

<u>Case (b)</u> $B = \begin{bmatrix} D & I_2 & & \Theta \\ & \ddots & \ddots & \\ & & & I_2 \\ \Theta & & & D \end{bmatrix}$ $D = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$B = \begin{bmatrix} D & & \Theta \\ & \ddots & \\ \Theta & & D \end{bmatrix} + \begin{bmatrix} 0 & I_2 & & 0 \\ & \ddots & \ddots & \\ & & 0 & I_2 \\ & & & 0 \end{bmatrix} = S + N$$

$$e^{Bt} = e^{St} \cdot e^{Nt}$$

$$= \begin{bmatrix} e^D & & 0 \\ & \ddots & \\ 0 & & e^D \end{bmatrix} \cdot \begin{bmatrix} I_2 & I_2 t & \cdots & \frac{(I_2 t)^{m-1}}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & I_2 t \\ & & & I_2 \end{bmatrix}$$

$$e^D = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} = e^{at} \cdot R$$

rotation matrix

$$e^{Bt} = e^{at} \begin{bmatrix} R & Rt & \cdots & \frac{(Rt)^{m-1}}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & Rt \\ & & & R \end{bmatrix}$$

To construct Jordan form of A, we need to get the <u>Jordan canonical basis</u> of generalized eigenvectors.

Denote $V_k = \ker A^k$, $k = 0, 1, \dots m$

Assume $A^m = \Theta$, $A^{m-1} \neq \Theta$, assume we are taking $\lambda_1 = 0$.

$$\{\Theta\} = V_0 \subset V_1 \subset \dots \subset V_m = V$$

$$\ker A = \{x \in V \,/\, Ax = 0\} \qquad V = \ker A \oplus \operatorname{im} A$$

$V_m = V$		
$V_{m-1} \oplus V'_{m-1} = V_m$	V'_{m-1}	$e_1, \dots e_s$
$V_{m-2} \oplus V'_{m-2} = V_{m-1}$	V'_{m-2}	$Ae_1, \dots Ae_s, e_{s+1}, \dots e_r$
$\vdots$		$\vdots$
$\Theta = V_0 \oplus V'_0 = V_1$	V'_0	$A^{m-1}e_1, \dots A^{m-1}e_s, A^{m-2}e_{s+1} \dots A^{m-2}e_r,$
		$A^{m-3}e_{r+1}, \dots -$

Jordan basis

<u>Ex.</u> $n = 2$

$$J = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$n = 3$

$$J = \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{bmatrix}, \quad J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\operatorname{span}\{v_1, v_2, v_3\} = \mathbb{R}^3 \quad = \operatorname{span}\{v_1, v_2, w_3\} \quad = \operatorname{span}\{v_1, w_2, w_3\}$$

Suppose $p_A(\lambda) = (\lambda-1)^2(\lambda+1)^2$

1. let $\lambda=1$ give $\dim(\ker(A-I))=1$

$\Rightarrow \delta_1 = 1$ is the number of Jordan blocks

$\ker(A-I) \subsetneq \ker(A-I)^2$

(a) Find $v \in \ker(A-I)^2 \setminus \ker(A-I)$

(b) $\{v, Av\}$ – corresp. Jordan basis contribution

2. let $\lambda=-1$ give $\dim(A+I)=2$

$\Rightarrow \{v_1, v_2\}$ – 2 indep. eigenvectors

$\Rightarrow$ put them into the Jordan basis

Ex. 1.

$$A=\begin{pmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -2 & -1 & -1 \\ 1 & 2-\lambda & 1 & 1 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & -2 & -1 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$=(1-\lambda)\big((1-\lambda)(\lambda^2-2\lambda+2)-(-\lambda+1)\big)$$

$$=(1-\lambda)\big((-\lambda)(\lambda^2-2\lambda+1)\big) = (\lambda-1)^4$$

$$A-\lambda I = \begin{bmatrix} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\delta_1 = \dim \ker (A-\lambda I) = 2 \Rightarrow 2$ eigenvectors

$x_2 = 0$

$x_1 + x_3 + x_4 = 0$

$$V_1^{(1)} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad V_2^{(1)} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$x_3 = 1$, $x_4 = 0$ $\qquad$ $x_3 = 0$, $x_4 = 1$

$$(A-\lambda I)v = c_1 V_1^{(1)} + c_2 V_2^{(1)}$$

$$\begin{bmatrix} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ 0 \\ c_1 \\ c_2 \end{bmatrix}$$

$x_2 = c_1$

$x_1 + x_2 + x_3 + x_4 = 0$

$c_2 = 0$

$$\xcancel{\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix}}$$

pick $\underline{c_1 = 1}$

$\boxed{\delta_2 = 3}$ $\qquad V_1^{(2)} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad$ corresp. to $v_1^{(1)} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\| \quad \dim \ker (A-\lambda I)^2$ spanned by $\{V_1^{(1)}, V_1^{(2)}, V_2^{(1)}\}$

$\boxed{\delta_3 = 4} = \dim \ker (A-\lambda I)^3 \quad : \quad (A-\lambda I) V_3^{(3)} = V_1^{(2)}$

$$\begin{bmatrix} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 + x_4 = 1$

$x_2 = 0$

$$v_1^{(3)} = \cancel{\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow P = \cancel{v_1^{(1)}, v_1^{(2)},} \ \{v_2^{(1)}, v_1^{(1)}, v_1^{(2)}, v_1^{(3)}\}$$ – Jordan Basis